VISCOSITY SOLUTIONS OF OPTIMAL STOPPING PROBLEMS FOR FELLER PROCESSES AND THEIR APPLICATIONS

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I would like to dedicate this thesis to my loving family and caring supervisor.

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Abstract

This thesis constitutes a research work on deriving viscosity solutions to optimal stopping problems for Feller processes. We present conditions on the process under which the value function is the unique viscosity solution to a Hamilton-Jacobi-Bellman equation associated with a particular operator. More specifically, assuming that the underlying controlled process is a Feller process, we prove the uniqueness of the viscosity solution. We also apply our results to study several examples of Feller processes. On the other hand, we try to extend our results by iterative optimal stopping methods in the rest of the work. This approach gives a numerical method to approximate the value function and suggest a way of finding the unique viscosity solution associated to the optimal stopping problem. We use it to study several relevant control problems which can reduce to corresponding optimal stopping problems. e.g., an impulse control problem as well as an optimal stopping problem for jump diffusions and regime switching processes. In the end, as a complementary, we are trying to construct optimal stopping problems with multiplicative functionals related to a non-conservative Feller semigroup. As a consequence, viscosity solutions were obtained for such kind of constructions.

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Chapter 1

Introduction

1.1 An Overview of Optimal Stopping Theory

It is intriguing to study the question of decision making and especially when to take a particular action in order to achieve one's best benefits in reality. This is known as the optimal stopping problem. Mathematically speaking, optimal stopping problems can be seen as the problem of computing a stopping time such that the expected payoff is maximized. Problems of optimal control under a stochastic framework were traditionally studied using dynamic programming equation (see for example Bellman [2013]). Specifically, the general optimal stopping problems can be formulated as:

Let $\mathfrak{T} \subseteq [0, \infty)$ and $\{X(t)\}_{t\in\mathfrak{T}}$ be a stochastic process with a state space E defined on $(\Omega, \mathsf{F}, \{\mathsf{F}_t\}_{t\in\mathfrak{T}})$. We aim to find an $\{\mathsf{F}_t\}_{t\in\mathfrak{T}}$-stopping time τ^* such that

$$\boldsymbol{E}\big[X_{\tau^*}\big] = \sup_{\tau} \boldsymbol{E}\big[X_\tau\big].$$

Considering the case $\mathfrak{T} = \mathbb{N}$, Snell proved that the optimal stopping time is

$$\tau^* = \inf\{n \in \mathbb{N}; X_n = Y_n\},$$

where $\{Y_n\}_{n\in\mathbb{N}}$ is the minimal regular supermatingale dominating $\{X_n\}_{n\in\mathbb{N}}$, that is,

$$Y_n := ess \sup_{\tau \geq n} \boldsymbol{E}\big[Y_\tau | \mathsf{F}_n\big],$$

which is refereed to as Snell envelope (see for example Peskir and Shiryaev [2006a]).

In this thesis, we focus on the infinite time horizon optimal stopping problems where $\mathfrak{T} = [0, \infty)$. A specific formulation is as follows

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \right],$$

where f and g are real valued measurable functions. For example, the perpetual American put option is a contract that the holder can exercise the right to sell the share at any time at a pre-determined price K. In this case, the share price can be modelled as a stochastic process, and f is the continuous cost paid by the holder and g is the payoff obtained when the holder exercises the right to sell the share, that is, $g(x) = (K - x)^+$. The holder has to make a decision when to exercise the right to sell the share (based on the share price) to maximize his discounted reward achieved from this option contract.

Generally, there are two ways of solving such optimal stopping problems. One approach relies on obtaining explicit solutions from free boundary problems. This setting can be widely applied to problems with multidimensional diffusion processes. The other approach is based on martingale technique from Snell envelope.

Alternatively, this thesis is interested in using viscosity solutions to characterize the value function of optimal stopping problems. A traditional approach is to derive the value function by assuming the solution is sufficiently smooth based upon verification theorem. However, this does not in most case take place. Rather the value function is normally a viscosity solution with weakened regularity assumptions. Therefore, in order to show that viscosity solution is the value function, the uniqueness of the viscosity should be proved then.

For example, consider a second-order partial differential equation of the form

$$F(x, u(x), D_x u(x), D_{xx} u(x)) = 0 \text{ for } x \in \mathcal{O} \tag{1.1.1}$$

where $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$ and $\mathcal{O} \subseteq \mathbb{R}^n$. Its viscosity solution is defined by

Definition 1.1. *A function $u \in \mathcal{C}(\mathcal{O})$ is a viscosity subsolution (respectively, supersolution) to* (1.1.1) *if for any $x_0 \in \mathcal{O}$ and $\phi \in \mathcal{C}^2(\mathcal{O})$ such that $u - \phi$ has local maximum (respectively, minimum) at x_0 in $\mathcal{O}$,*

$$F(x_0, \phi(x_0), D_x\phi(x_0), D_{xx}\phi(x_0)) \leq (\geq) 0,$$

Crandall et al. [1992] shows the existence, uniqueness and stability of the viscosity solution. The main contribution in this thesis are in three folds: (1) we replace the operator D_x and D_{xx} by an infinitesimal generator of the associated Markov process, the domain $\mathcal{C}^2(\mathcal{O})$ by the domain of the infinitesimal generator and state space $\mathcal{R}^d$ by any metric space E. At the same time, the existence and uniqueness of the viscosity solution will still be discussed with more in in Chapter

3. The main contribution in this Chapter is that we prove the uniqueness of the viscosity solution of Feller process using Hille-Yosida theorem. As far as we are concerned, this approach is quite novel in literature.

1.2 Optimal stopping problems for Feller processes

Optimal stopping problems for Markov processes have been extensively studied in the literature using different approaches. Such problems are very important due to their various applications in engineering, physics, mathematical finance and insurance. See for example Peskir and Shiryaev [2006b] in which different methods to solve optimal stopping problems are given. Assuming that the state process is given by a diffusion process (with non degenerate diffusion coefficient), the pioneering book [Bensoussan and Lions, 1978, Chapter 3] introduces a variational inequality approach to solve optimal stopping problems. Under some weak regularity of the data the authors prove the regularity of the value function. Since then, there have been many studies on optimal stopping problems for Markov processes using the variational inequality approach, with the aim of relaxing the assumptions on the class of Markov processes and/or the assumptions on the reward functional as well as looking at the properties of the value function. Note however that the associated variational inequality to the optimal stopping problem is often difficult to solve, unless one allows a notion of weak solution, called viscosity solution, to the Hamilton-Jacobi-Bellman (HJB) equation. In the case of a diffusion process, this approach is used for example in Bassan and Ceci [2002a,b]; Fleming and Soner [2006]; see also Øksendal and Sulem [2007] for the jump-diffusion case.

In studying the viscosity properties of the value function, the traditional approach assumes that the generator associated with the state Markov process is given by parabolic or elliptic differential operators. Hence, one can use tools from partial differential equations to solve the problem. A natural question is what happens when the state process is given by a Markov process (for example a Feller process) for which the generator is not given by a partial differential operator but only derived from its semigroup. To the best of our knowledge, only Palczewski and Stettner [2014] deals with existence of viscosity solution of an HJB equation when the generator is derived from a Feller semigroup.

One of the main motivation of this thesis is to provide a general analytical approach that extends earlier results on properties of the value function to a more general class of processes. We will not always assume that the generator of the process is given by a partial differential operator. The other motivation is to establish a framework that enables to find the value function of an optimal stopping problem for a general class of processes (Feller processes) by analytically deriving the unique viscosity solution to the associated HJB equation. In this regards, our result completes the previous studies, in the sense that, the existence of viscosity solutions to the HJB equation is known (see Palczewski and Stettner [2014]) and the uniqueness of the viscosity solution was only conjectured. To our knowledge, we do not know of any existing results on uniqueness of viscosity solution in this framework.

To be more precise, in Chapter 3, we consider an infinite time horizon optimal stopping problems with fixed discount rate using the penalty method introduced in Robin [1978] and the general setting in Stettner and Zabczyk. Contrary to the traditional method, based on calculations of the (integro) differential operators, this method is based on an efficient approximation of the value function by smooth functions. Although there are several extensions of the penalty method (see for example Palczewski and Stettner [2010, 2011, 2014]; Stettner [2011]), most of them only focus on studying the continuity of the value function except work Palczewski and Stettner [2014] which studies the existence of viscosity solution to the associated HJB equation. In this thesis, under slightly different conditions, we show that the value function is the unique viscosity solution to the HJB equation associated with the optimal stopping problem.

We apply our result to study viscosity properties of the value function for optimal stopping problems of Lévy processes, reflected Brownian motion, sticky Brownian motion, diffusion with piecewise coefficients and semi-Markov processes. We show that depending on the choice of the operator and its domain, the value function is the unique viscosity solution associated with the HJB equation. Let us mention that our viscosity analysis on diffusion with piecewise coefficients and semi-Markov processes are typically not investigated in the current literature on optimal stopping problems. In the former case, we will see later (confer Corollary 3.46 and Corollary 3.47) that the value function is a viscosity solution associated with a particular operator to an HJB equation. In the latter case, we first use perturbation theory (confer Böttcher et al. [2013]) to transform the one-dimensional semi-Markov process to a two-dimensional Markov process. Then, we show that the value function of the problem is the unique viscosity solution to the associated HJB equation. Similar optimal stopping problem was studied in Boshuizen and Gouweleeuw [1993]; Muciek [2002] using iterative approach. We also use our results to explicitly derive the value function and the optimal stopping time in the case of a straddle option for the following state processes: reflected Brownian motion (see Corollary 3.54); Brownian motion with jump at boundary (see Proposition 3.55) and regime switching Feller diffusion (see Corollary 3.59).

1.3 Iterative optimal stopping method

The iterative method for impulse control problems was first introduced by Bensoussan [1984], assuming that the state process is given by a diffusion process. The idea was to reduce the quasi-variational inequality to a sequence of variational inequaility. Such reduction for combining optimal control and optimal stopping was studied by Chancelier et al. [2002]. Similar types of reduction results can be also found in Hdhiri and Karouf [2011]; Øksendal and Sulem [2002, 2007]; Seydel [2009].

On the contrary, Robin [1978] studied the regularity of the value function for impulse control problem using the iterative optimal method, where the underlying process is a Feller process. The motivation of Chapter 4 is from the fact discussed in Chapter 3, where we analysed the optimal stopping problem and characterized its value function by the unique viscosity solution to

$$\min(aw - \mathcal{A}w - f, w - g) = 0, \tag{1.3.1}$$

where $\mathcal{A}$ is a generator derived from some semigroup. Since most of the impulse control problems can be reduced to an iterative optimal stopping problem, we are able to extend the results in Chapter 3 to a more general case (see Section 4.3).

However, here we do not restrict our results to impulse control problems only. Our main contirbution is to solve problems under a more general setting including processes constructed by perturbations and optimal stopping problems without discount. Furthermore, in Chapter 3, the viscosity solutions are derived via a method using generators formed by a semigroup, instead of the traditional way of applying the approach developed in Crandall et al. [1992]. Thus, in Chapter 4 we are able to extend the method to solve for more abstract cases. More precisely, we establish the existence of the viscosity solution to

$$\min(aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w) = 0, \tag{1.3.2}$$

where F and G are abstract operators on $\mathcal{C}_b(\mathsf{E})$. For example, similar formulation can be found for impulse control problems given by [Zabczyk, 2009, Chapter 2.] for deterministic processes as well as [Øksendal and Sulem, 2007, Chapter 8.] (for jump diffusion, where the operator G defined by

$$\mathsf{G}u(x) := \sup_{y \in \mathsf{E}}(u(y) + K(x,y)),$$

where $K : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$.

One contribution of this work is to relax the specific operator (for example G) to an abstract operator with monotonicity and convexity properties on F and

G. Therefore, we consider the related problems for more general Markov Feller processes.

In fact, iterative optimal stopping methods have just recently evolved in literature. For instance, Le and Wang [2010] analysed the properties of the solution of a finite time optimal stopping (American) option pricing problem under regime switching by iterative optimal stopping method. A similar approach has also been seen in Babbin et al. [2014]. Apart from the regime switching case, we can also find the iterative optimal stopping method being applied in jump diffusions according to Bayraktar and Xing [2009]. Rather than separating the above issues, we combine both cases by incorporating perturbations into Feller processes.

Added to this, we will show that the underlying approach can be employed to work with optimal stopping problems without discount. As far as we are concerned, there have not been any research pursuing this direction based on this method.

Even though the iterative optimal stopping method has been used to study the aforementioned two issues, its usage was analysed case by case. However, we are able to summarise common features from all scenarios studied in literature, and demonstrate a general assumption under which the underlying problems could be dealt with in a generalised manner.

1.4 Optimal stopping problems with multiplicative functionals

In Chapter 5, we are concerned with the optimal stopping problem

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-as} M(s) f(X(s)) \mathrm{d}s + e^{-a\tau} M(\tau) g(X(\tau)) \right],$$

where M is the corresponding multiplicative functional of X. Preliminaries of the multiplicative functionals will be introduced in Section 2.3, Chapter 2. 5 is aimed at constructing optimal stopping problems whose viscosity solutions are associated with generators of non-conservative Feller semigroups, whereas the conservative ones are well explained in Chapter 3. We found that they relate to a series of optimal stopping problems with multiplicative functionals. Two common problems are mainly illustrated and solved in this chapter. One is an optimal stopping problem with a random discount, i.e., when $M(t) = e^{\int_0^t r(X(s))\mathrm{d}s}$,

$$V(x) = \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-as} e^{\int_0^s r(X(z))\mathrm{d}z} f(X(s)) \mathrm{d}s + e^{-a\tau} e^{\int_0^{\tau} r(X(z))\mathrm{d}z} g(X(\tau)) \right], \tag{1.4.1}$$

where $r \in \mathcal{C}_b(\mathsf{E})$ is positive. The other is an optimal stopping problem until hitting time $\tau_{\mathcal{O}} := \inf\{s \geq 0; X(s) \notin \mathcal{O}\}$, where $\mathcal{O}$ is the subset of E, i.e., when $M(t) = \mathbf{1}_{t<\tau_{\mathcal{O}}}$,

$$V(x) = \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau\wedge\tau_{\mathcal{O}}} e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau} g(X(\tau))\mathbf{1}_{\tau<\tau_{\mathcal{O}}} \right]. \tag{1.4.2}$$

These two problems can be straightforwardly solved as a standard optimal stopping problem without multiplicative functionals. However, the idea of multiplicative functional is not considered in general optimal stopping cases. Beibel and Lerche [2001] worked out this optimal stopping problem with multiplicative functionals explicitly using martingale arguments. Cissé et al. [2012] extends this result to the one-sided regular Feller processes. Furthermore, optimal stopping problems with multiplicative functionals are also employed to study the regularity of the value functions for general Markov-Feller processes. In particular, Palczewski and Stettner [2014] took into account of the optimal stopping problems (1.4.1) with discount rates which are not uniformly separated from 0. For the case of (1.4.2), Robin [1978] used the penalty method to analyse properties of its value function. Recently, Stettner [2011] extends that result over a finite time horizon. That is to say, $\mathcal{O} = [0, T) \times \mathsf{E}_0$ while the Feller process has a form (D, X) with state space $\mathsf{E} = [0, \infty) \times \mathsf{E}_0$ where $D_t := D_0 + t$ is the time horizon. We should also mention that Palczewski and Stettner [2011] considers a case where g can be discontinuous at the boundary $\partial\mathcal{O}$.

However, these literature are all based on the penalty method with one specific question in each study. In Chapter 5, we use a different approach through finding equivalent optimal stopping problems rather than modifying the penalty methods. We notice that whether we can reduce an optimal stopping problem with multiplicative functionals to one without multiplicative functionals is based on the conditions of the multiplicative functionals themselves. Building on this idea, we can analyse the relevant optimal stopping problems in a general way borrowing the method of multiplicative functionals, instead of working with certain optimal stopping problems case by case.

The rest of the thesis will be organised as follows. Chapter 2 describes some preliminary knowledge of Feller processes and multiplicative functionals. Chapter 3 shows the viscosity solution to an optimal stopping problems for Feller processes. Inspired by this, we extend our problems using iterative optimal stopping method in Chapter 4. It can be applied to deal with impulse control problems, optimal stopping problems with perturbation and optimal stopping problem with zero discount. Chapter 5 briefly discusses the optimal stopping problem with multiplicative functionals. A list of bibliography can be found at the end of the thesis.

Chapter 2

Preliminaries

This chapter serves as a foundation of the work to be presented in this thesis. We first present some basic definitions and properties of Feller processes and Feller semigroups. Then, we formulate the optimal stopping problems discussed in this chapter and introduce our main assumptions. For more information on Feller processes, the reader may consult for example [Kallenberg, 2006, Chapter 17] or [Böttcher et al., 2013, Chapter 1].

2.1 Notations

Throughout this thesis, we suppose that E is a locally compact, separable metric space with metric ρ. $\mathcal{E}$ is the σ-algebra of the Borel sets of E. If E is not compact, we define $\mathsf{E}_\partial := \mathsf{E} \cup \{\partial\}$ as the one point (Alexandorff) compactification of E, where $\{\partial\}$ is the point at infinity. Otherwise, $\{\partial\}$ is an isolated point from E. In both cases, E_∂ is compact and metrizable and $\mathcal{E}_\partial$ denotes the σ-algebra in E_∂ generated by $\mathcal{E}$. The following notations will show up in this thesis.

- $B(\mathsf{E})$ is the space of all bounded Borel measurable functions on E;
- $\mathcal{C}(\mathsf{E})$ is the space of all continuous functions on E;
- $\mathcal{C}_c(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ has compact support}\}$;
- $\mathcal{C}_0(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ vanishes at infinity}\}$;
- $\mathcal{C}_*(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ converges at infinity}\}$;
- $\mathcal{C}_b(\mathsf{E}) := \mathcal{C}(\mathsf{E}) \cap B(\mathsf{E})$;
- $C^n(\mathcal{O})$ is the space of all the nth order differentiable equations;

- $USC(\mathsf{E})$ (*respectively*, $LSC(\mathsf{E})$) denotes the space Borel-measurable upper (*respectively*, lower) semicontinuous function on E;
- $\mathbb{R}$ is the set of real numbers;
- $\mathbb{R}^+ := [0, \infty)$;
- $\mathbb{N} := \{1, 2, 3, \ldots\}$,
- $\mathbf{1}_A$ is the indicator function of a set A;
- $u|_{\mathcal{O}}$ is the restriction of a function u to $\mathcal{O}$;
- $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$;
- $x^+ = \max(x, 0)$,
- $\boldsymbol{P}^x := \boldsymbol{P}[\cdot|X(0) = x]$ and $\boldsymbol{E}^x := \boldsymbol{E}[\cdot|X(0) = x]$.

Remark 2.1. *The above definitions imply that* $\mathcal{C}_c(\mathsf{E}) \subseteq \mathcal{C}_0(\mathsf{E}) \subseteq \mathcal{C}_*(\mathsf{E}) \subseteq \mathcal{C}_b(\mathsf{E})$. *Moreover, if* E *is compact, these spaces coincide.*

Let $\|\cdot\|_\infty$ be the supremum norm that is for any $w \in B(\mathsf{E})$,

$$\|f\|_\infty := \sup_{x \in \mathsf{E}} |f(x)|.$$

Equipped with this norm, $(\mathcal{C}_0(\mathsf{E}), \|\cdot\|_\infty)$, $(\mathcal{C}_*(\mathsf{E}), \|\cdot\|_\infty)$ and $(\mathcal{C}_b(\mathsf{E}), \|\cdot\|_\infty)$ are all the Banach spaces. The relation " $\leq$ " is a partial order on the space of real valued functions on E and we have $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathsf{E}$.

2.2 Feller Processes and Feller Semigroups

In this section, we give a series of definitions with respect to Feller semigropus as well as their related components.

Definition 2.2. *(Feller Semigroup) A collection of bounded linear operators* $\{\mathcal{P}_t\}_{t\geq 0}$ *is called Feller semigroups on* $\mathcal{C}_0(\mathsf{E})$*, if it satisfies the following four properties:*

- $\mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s$*, for all* $t, s \geq 0$*;* $\mathcal{P}_0 = \mathcal{I}$*, where* $\mathcal{I}$ *is the identity operator.*
- *For each* $t \geq 0$*, if* $w \in \mathcal{C}_0(\mathsf{E})$*,* $0 \leq w \leq 1$*, then,* $0 \leq \mathcal{P}_t w \leq 1$.
- *(Feller Property)* $\mathcal{P}_t : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ *for all* $t \geq 0$.
- *(Strong Continuous Property)* $\lim_{t\to 0^+} \|\mathcal{P}_t w - w\|_\infty = 0$ *for* $w \in \mathcal{C}_0(\mathsf{E})$.

Furthermore, a semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ *is conservative if* $\mathcal{P}_t 1 = 1$ *for all* $t \geq 0$.

Definition 2.3. (Feller Process) *A Feller process* $\{X(t)\}_{t\geq 0}$ *is a Markov process whose transition semigroup defined by*

$$\mathcal{P}_t w(x) := \boldsymbol{E}^x \left[w(X(t)) \right] \quad \text{for any } x \in \mathsf{E} \text{ and } w \in B(\mathsf{E})$$

is a Feller semigroup.

Based on Definition 2.3, the transition semigroup of a Feller process is conservative.

Definition 2.4. (Infinitesimal Generator) *An infinitesimal generator of a Feller semigroup* $\{\mathcal{P}_t\}_{t\geq 0}$ *or a Feller process* $\{X(t)\}_{t\geq 0}$ *is a linear operator* $(\mathcal{L}, D(\mathcal{L}))$, *with* $\mathcal{L} : D(\mathcal{L}) \subseteq \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ *defined by*

$$\mathcal{L}w := \lim_{t\to 0^+} \frac{\mathcal{P}_t w - w}{t} \quad \text{for } w \in D(\mathcal{L}), \tag{2.2.1}$$

where the domain

$$D(\mathcal{L}) := \left\{ w \in \mathcal{C}_0(\mathsf{E}); \text{the limit in (2.2.1) exists in } \mathcal{C}_0(\mathsf{E}) \right\}.$$

Definition 2.5. (Resolvent) *A resolvent* $\{\mathcal{R}_\lambda\}_{\lambda>0}$ *is defined by*

$$\mathcal{R}_\lambda w(x) := \int_0^\infty e^{-\lambda t} \mathcal{P}_t w(x) \, \mathrm{d}t \quad \text{for } x \in \mathsf{E} \text{ and } w \in \mathcal{C}_0(\mathsf{E}).$$

It is well known that the following resolvent identity equation is satisfied: for any $\lambda, \mu > 0$ and $w \in \mathcal{C}_0(\mathsf{E})$

$$\mathcal{R}_\lambda w - \mathcal{R}_\mu w = (\mu - \lambda)\mathcal{R}_\lambda \mathcal{R}_\mu w. \tag{2.2.2}$$

Definition 2.6. (*Postive Maximum Principle*) *An operator* $(\mathcal{L}, D(\mathcal{L}))$ *satisfies positive maximum principle if* $\mathcal{L}w(x_0) \leq 0$ *for any* $w \in D(\mathcal{L})$ *with* $w(x_0) = \sup_{x\in\mathsf{E}} w(x) \geq 0$.

We now states the Hille-Yosida-Ray theorem for strongly continuous semigroups. This theorem gives the relationships among Feller semigroup, generator and resolvent (see [Böttcher et al., 2013, Theorem 1.30]) and will play a key role in this thesis.

Theorem 2.7. *Let* $(\mathcal{G}, D(\mathcal{G}))$ *be a linear operator on* $\mathcal{C}_0(\mathsf{E})$. $(\mathcal{G}, D(\mathcal{G}))$ *is closable and its closure* $(\mathcal{G}, D(\mathcal{G}))$ *is the infinitesimal generator of a Feller semigroup if and only if*

1. *$D(\mathcal{G})$ is dense in $\mathcal{C}_0(\mathsf{E})$.*

2. *The range of $\lambda - \mathcal{G}$ is dense in $\mathcal{C}_0(\mathsf{E})$ for all $\lambda > 0$.*

3. *$(\mathcal{G}, D(\mathcal{G}))$ satisfies the positive maximum principle.*

The corollary below directly follows the Hille-Yosida theorem (see for example [Taira, 2004, Proposition 4.9 and Theorem 4.10]).

Corollary 2.8. *Let $(\mathcal{L}, D(\mathcal{L}))$ be the infinitesimal generator of some Feller semigroup. Then,*

1. *$(\mathcal{L}, D(\mathcal{L}))$ is closed,*

2. *For each $\lambda > 0$, the operator $(\lambda - \mathcal{L})$ is a bijection of $D(\mathcal{L})$ onto $\mathcal{C}_0(\mathsf{E})$ and its inverse is the resolvent $\mathcal{R}_\lambda$, that is, for all $w \in \mathcal{C}_0(\mathsf{E})$ and $v \in D(\mathcal{L})$, we have*
$$(\lambda - \mathcal{L})\mathcal{R}_\lambda w = w \ \ \textit{and} \ \ \mathcal{R}_\lambda(\lambda - \mathcal{L})v = v. \tag{2.2.3}$$

3. *For each $\lambda > 0$, we have the inequality*
$$\|\mathcal{R}_\lambda\|_\infty := \sup_{w \in \mathcal{C}_0(\mathsf{E})} \frac{\|\mathcal{G}_\lambda w\|_\infty}{\|w\|_\infty} \leq \frac{1}{\lambda} \tag{2.2.4}$$

Subsequently, we give the definition of the core, which will enable us to uniquely characterize a Feller semigroup.

Definition 2.9. *(Core) $(\mathcal{G}, D(\mathcal{G}))$ is called a core of an infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$ if it is a linear closable operator which satisfies $D(\mathcal{G}) \subseteq D(\mathcal{L})$ is dense in $\mathcal{C}_0(\mathsf{E})$ and the closure of $(\mathcal{G}, D(\mathcal{G}))$ is $(\mathcal{L}, D(\mathcal{L}))$, that is, for any $w \in D(\mathcal{L})$, there exists a sequence $\{w_n\}_{n \in \mathbb{N}^+}$ in $D(\mathcal{G})$ such that*
$$\lim_{n \to \infty} (\|w_n - w\|_\infty + \|\mathcal{G} w_n - \mathcal{L} w\|_\infty) = 0.$$

By (1) in Corollary 2.8, it follows that the infinitesimal generator itself is the core of an infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$ of a Feller semigroup.

2.2.1 Lévy Processes

This part introduces several examples of Feller semigroups. The first example is Lévy processes.

Example 2.10. *(Lévy processes) Let* $\{\mu_t\}_{t\geq 0}$ *be a family of infinitely divisible probability measures on* $\mathbb{R}^d$*. Then define the semigroup by*

$$\mathcal{P}_t u(x) := \int_{\mathbb{R}^d} u(x+y)\mu_t(\mathrm{d}y).$$

Let X *be a Lévy process with state space* $\mathbb{R}^d$ *and with the following properties*

1. *Stationary increments:* $X(t)-X(s)$ *has the same distribution with* $X(t-s)$*;*
2. *Independent increments: for* $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n < \infty$, $X(t_2)-X(t_1)$, $X(t_3)-X(t_2)$, $\cdots$, $X(t_n)-X(t_{n-1})$ *are independent;*
3. *Stochastic continuity:* $\lim_{h\to 0} \boldsymbol{P}(|X(h)| > \varepsilon) = 0$.

Assume that $\boldsymbol{P}^x\big[X(t) \in \mathrm{d}y\big] = \mu_t(\mathrm{d}y - x)$*, the Feller semigroup* $\{\mathcal{P}_t\}_{t\geq 0}$ *can be alternatively defined by*

$$\mathcal{P}_t u(x) = \boldsymbol{E}^x\big[u(X(t))\big].$$

Its corresponding generator of the form

$$\begin{aligned} &D(\mathcal{L}) := \mathcal{C}_0^2(\mathbb{R}^d) \\ &\mathcal{L}u(x) := l\cdot\nabla u(x) + \frac{1}{2} div Q\nabla u(x) + \int_{\mathbb{R}^d\setminus 0} (u(x+y) - u(x) - \nabla u(x)\cdot y\mathcal{X}(|y|))\nu(\mathrm{d}y), \end{aligned}$$

where $l \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ *positive semidfinite and* ν *is a positive Radon measure satisfying* $\int_{\mathbb{R}^d\setminus 0} \min(|y|^2, 1)\nu(\mathrm{d}y) < \infty$ *and* $\mathcal{X} \in B(\mathbb{R}^+)$ *satisfies* $0 \leq 1 - \mathcal{X}(s) \leq \kappa \min(s, 1)$ *for some* $\kappa > 0$ *and* $s\mathcal{X}(s)$ *is bounded.*

Based up Lévy processes, we can construct several examples which are also Feller processes.

Example 2.11. *(Ornstein-Uhlenbeck semigroups) Let* $\{\mu_t\}_{t\geq 0}$ *be a family of infinitely divisible probability measures on* $\mathbb{R}^d$ *such that* $t \to \mu_t$ *is continuous in the vague topology. Then the Ornstein-Uhlenbeck semigroups is defined by*

$$\mathcal{P}_t u(x) := \int_{\mathbb{R}^d} u(e^{tB}x + y)\mu_t(\mathrm{d}y),$$

where $B \in \mathbb{R}^{d\times d}$*. Furthermore, let* $\{Z(t)\}_{t\geq 0}$ *be a Lévy process with* $\{\mu_t\}_{t\geq 0}$*. The process*

$$X_t^x := e^{tB}x + \int_0^t e^{(t-s)B}\mathrm{d}Z(s)$$

for $x \in \mathbb{R}^d$ is a Ornstein-Uhlenbeck process, which is the Feller process (see for example [Sato and Yamazato, 1984, Theorem 3.1]).

[Böttcher et al., 2013, Theorem 3.8] also shows a general case constructed by multidimensional Lévy processes.

Theorem 2.12. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ be a locally Lipschitz continuous and bounded, and let $\{L(t)\}_{t\geq 0}$ be an n-dimensional Lévy process. Then, the solution of the stochastic differential equation*

$$dX(t) = \Phi(X(t_-))\mathrm{d}L(t)$$

exists for every initial condition $X(0) = x \in \mathbb{R}^d$ and yields a Feller process.

We should emphasis that here Φ should be bounded to make sure the solution is a Feller process.

2.2.2 Examples of Infinitesimal Generator

Now we present several infinitesimal generators of Feller processes.

Example 2.13.

1. *(Uniform motion) Let $v \in \mathcal{C}_b(\mathbb{R})$.*

$$\begin{aligned} D(\mathcal{L}) &:= \{u \in \mathcal{C}_0^1(\mathbb{R}); D_x u(x) \in \mathcal{C}_0(\mathbb{R})\} \\ \mathcal{L}u(x) &:= v(x) D_x u(x). \end{aligned}$$

2. *(Poisson processes)Let $\lambda \in \mathbb{R}$ be strictly positive.*

$$\begin{aligned} D(\mathcal{L}) &:= \mathcal{C}_0(\mathbb{R}) \\ \mathcal{L}u(x) &:= \lambda(u(x+1) - u(x)). \end{aligned}$$

3. *(Brownian motion with a constant drift)*

$$\begin{aligned} D(\mathcal{L}) &:= \{u \in \mathcal{C}_0^2(\mathbb{R}); D_x u(x) \in \mathcal{C}_0(\mathbb{R}), D_{xx} u(x) \in \mathcal{C}_0(\mathbb{R})\} \\ \mathcal{L}u(x) &:= \frac{1}{2}\sigma^2 D_{xx} u(x) + \mu D_x u(x). \end{aligned}$$

4. *(Reflecting Brownian motion)*

$$\begin{aligned} D(\mathcal{L}) &:= \{u \in \mathcal{C}_0^2(\mathbb{R}^+); D_x u(x) \in \mathcal{C}_0(\mathbb{R}^+), D_{xx} u(x) \in \mathcal{C}_0(\mathbb{R}^+), D_x u(x) = 0\} \\ \mathcal{L}u(x) &:= \frac{1}{2} D_{xx} u(x). \end{aligned}$$

5. *(Sticking barrier Brownian motion)*

$$D(\mathcal{L}) := \{u \in \mathcal{C}_0^2(\mathbb{R}^+); D_x u(x) \in \mathcal{C}_0(\mathbb{R}^+), D_{xx}u(x) \in \mathcal{C}_0(\mathbb{R}^+), D_{xx}u(x) = 0\}$$
$$\mathcal{L}u(x) := \frac{1}{2}D_{xx}u(x).$$

6. *(Absorbing barrier Brownian motion)*

$$D(\mathcal{L}) := \{u \in \mathcal{C}_0^2((0,\infty)); D_x u(x) \in \mathcal{C}_0((0,\infty)), D_{xx}u(x) \in \mathcal{C}_0((0,\infty)), u(x) = 0\}$$
$$\mathcal{L}u(x) := \frac{1}{2}D_{xx}u(x).$$

Example 2.14. *(Feller diffusion) Assume that $\mathsf{E} = \mathbb{R}^n$ and define the life time of X by $\xi = \{t \geq 0; X(t) = \partial\}$. Feller diffusion is a a Feller process which has a continuous path $t \to X(t)(\omega)$ on $[0,\xi)$ whose domain of the generator contains $\mathcal{C}_c^\infty(\mathbb{R}^n)$. It can be proved (see [Kallenberg, 2006, Theorem 17.24]) that the restriction $(\mathcal{G}^{FD}, D(\mathcal{G}^{FD}))$ of the infinitesimal generator of the Feller diffusion X on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is of the form as follows*

$$\mathcal{G}^{FD}u(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}u(x) - v(x)u(x), \tag{2.2.5}$$

where $a_{ij}, b_i, v \in \mathcal{C}(\mathbb{R}^n)$, $v \geq 0$ and $\{a_{ij}(x)\}_{i,j}$ is non-negative symmetric matrix for all $x \in \mathsf{E}$.

2.2.3 Transformations of Feller Processes

Markov processes can be transformed in many ways. [Böttcher et al., 2013, Chapter 4] introduced several transformations of Feller generators. Examples will be provided along with relevant transformations later on in this thesis.

Theorem 2.15. *[Böttcher et al., 2013, Theorem 4.1 and Corollary 4.2] Let $(\mathcal{L}, D(\mathcal{L}))$ be a generator of the Feller process $\{X(t), \mathcal{F}_t\}_{t\geq 0}$ and $s \in \mathcal{C}_b(\mathbb{R}^n)$ be real valued and strictly positive. Then the closure of $(s(\cdot)\mathcal{L}, D(\mathcal{L}))$ is also a Feller generator. Furthermore, denote by*

$$\alpha(t,\omega) := \int_0^t \frac{\mathrm{d}r}{s(X(r)(\omega))} \text{ and } \alpha(\infty,\omega) := \int_0^\infty \frac{\mathrm{d}r}{s(X(r)(\omega))}$$

and by $\tau(t)$ the inverse

$$\tau(t,\omega) := \inf\{u > 0 : \alpha(u,\omega) > t\}, \inf\emptyset = +\infty.$$

The process $\{\hat{X}(t), \hat{\mathcal{F}}_t\}_{t\geq 0}$ defined by $\hat{X}(t) := X_{\tau(t)}$ and $\hat{\mathcal{F}}_t := \mathcal{F}_{\tau(t)}$ is again a Feller process with the generator $(s(\cdot)\mathcal{L}, D(\mathcal{L}))$.

Another powerful tool is to transform a known Feller process to a new Feller process by perturbation. We first introduce the following lemma which enables us to construct the Feller semigroup using perturbations.

Lemma 2.16. *Böttcher et al. [2013] Let $(\mathcal{G}, D(\mathcal{G}))$ be the infinitesimal generator of some Feller semigroup on $\mathcal{C}_0(\mathsf{E})$. Assume that $B : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ and B is bounded, that is, there exists $C > 0$ such that $\sup_{u\in\mathcal{C}_0(\mathsf{E})} \frac{\|Bu\|_\infty}{\|u\|_\infty} \leq C$. Additionally suppose $(B, \mathcal{C}_0(\mathsf{E}))$ satisfies the positive maximum principle. Then, $(\mathcal{L}+B, D(\mathcal{L}))$ is the infinitesimal generator of some Feller semigroup on $\mathcal{C}_0(\mathsf{E})$.*

2.3 Multiplicative Functional and Subprocesses

In this section we present the fundamental properties of multiplicative functionals of a Markov process that are relevant to this thesis and serve as preliminaries for Chapter 5. For a more concrete explanation, readers are advised to check [Blumenthal and Getoor, 2007, Chapter III] and [Rogers and Williams, 2000, Chapter 18]. Further to previous settings, here we impose additional conditions in order to introduce multiplicative functionals.

1. ∂ is an absorbing state such that $X(t) = \partial$ for any $t \geq s$ if $X(s) = \partial$,
2. there is a distinguished point w_∂ in Ω such that $X(0)(\omega_\partial) = \partial$.
3. the life time of X is defined by $\eta_X := \inf\{t \geq 0; X(t) = \partial\}$.
4. the time horizon is extended to $\bar{\mathbb{R}}_+ := [0, \infty]$ such that $X(\infty)(\omega) = \partial$ and $\theta_\infty(\omega) = \omega_\partial$ for all $\omega = \Omega$.

Initially, a formal definition of a multiplicative functional is given in Definition 2.17.

Definition 2.17. *(Multiplicative functional) A family of real valued random variables $M = \{M(t)\}_{t\geq 0}$ on $(\Omega, \mathcal{F})$ is called multiplicative functional of X if it satisfies*

1. *$M(t)$ is $\mathcal{F}_t$-measurable for $t \geq 0$,*
2. *$M_{t+s} = M(t) \cdot \theta_t(M(s))$ for each $t, s \geq 0$,*
3. *$0 \leq M(t)(\omega) \leq 1$ for all $t \geq 0$ and $\omega \in \Omega$.*

Subsequently, let us look at its properties associated definitions.

Definition 2.18. *(Measurable, Right continuous and Permanent points)*

1. *A multiplicative functional M is measurable if the family $\{M(t)\}_{t\geq 0}$ is progressively measurable relative to $\{\mathcal{F}_t\}_{t\geq 0}$.*

2. *M is right continuous if $t \to M(t)$ is right continuous almost surely.*

3. *A point $x \in \mathsf{E}$ is called a permanent point for M if $\boldsymbol{P}^x(M(0) = 1) = 1$ and define E_M by the set of all the permanent points.*

Definition 2.19. *(Strong and Regular multiplicative functional)*

1. *M is called a strong multiplicative functional if it is right continuous and satisfies*

$$\boldsymbol{E}^x\big[u(X(t+\tau))M_{t+\tau})\big] = \boldsymbol{E}^x\big[\boldsymbol{E}^{X(\tau)}[u(X(t))M(t)]M(\tau)\big] \tag{2.3.1}$$

   for all $x \in \mathsf{E}$ and $\mathcal{F}_t$-stopping time τ.

2. *M is called a regular multiplicative functional if it is right continuous and satisfies*

$$\boldsymbol{P}^x\left[X(\tau) \in \mathsf{E} \setminus \mathsf{E}_M; M(\tau) > 0\right] = 0 \tag{2.3.2}$$

   for all $x \in \mathsf{E}$, $t \geq 0$, $w \in B(\mathsf{E})$ and $\mathcal{F}_t$-stopping time τ.

The above two properties actually play a major role of deducing results in Chapter 5. Intuitively, strong multiplicative functional resembles strong Markov property. These two definitions are thus interchangeable.

Lemma 2.20. *[Blumenthal and Getoor, 2007, Chapter III, Proposition 4.21] M is a strong multiplicative functional if and only if M is a regular multiplicative functional.*

Next, several famous examples are demonstrated below. Clearly, they are all multiplicative functionals as a direct result of applying Definition 2.17. The existence and uniqueness of the viscosity solution to the optimal stopping problems with these multiplicative functionals will be further discussed in Chapter 5.

Example 2.21.

1. *For $a \geq 0$, $M(t) = e^{-at}$ is a multiplicative functional with $\mathsf{E}_M = \mathsf{E}$.*

2. *Let X be progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and let $r \in \mathcal{C}_b(\mathsf{E})$ be a positive function. Define*

$$M(t) = e^{-\int_0^t r(X(s))\mathrm{d}s}.$$

3. *Let $\mathcal{O}$ be an open subset and a $\mathcal{F}_t$-stopping time $\tau_{\mathcal{O}} := \inf\{t \geq 0; X(t) \notin \mathcal{O}\}$. Define $M(t)(\omega) = \mathbf{1}_{t\in[0,\tau_{\mathcal{O}}(\omega))}$ for $\omega \in \Omega$.*

4. *A $\mathcal{F}_t$-stopping time ζ is a terminal time of X if*

$$\zeta = \tau + \zeta \circ \theta_t,$$

almost surely on $\{\zeta > \tau\}$. Define

$$M(t)(\omega) = \mathbf{1}_{\zeta(\omega)<t} \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{2.3.3}$$

As can be seen above, (1)-(3) are regular multiplicative functionals, but (4) is not necessarily the case. In order for (4) to be a regular multiplicative functional, we need to impose a strong terminal time as explained in Lemma 2.22.

Lemma 2.22. *(See [Blumenthal and Getoor, 2007, Page 124]) A $\mathcal{F}_t$-stopping time ζ is a strong terminal time of X if*

$$\zeta = \tau + \zeta \circ \theta_\tau,$$

for all $\mathcal{F}_t$-stopping time τ almost surely on $\{\zeta > \tau\}$. Define M^ζ by

$$M(t)^\zeta(\omega) = \mathbf{1}_{\zeta(\omega)<t} \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{2.3.4}$$

Then M is a regular multiplicative functional.

In what follows, we identify a connection between multiplicative functionals with Feller semigroups. Let $\{\mathcal{P}_t\}_{t\geq 0}$ be a Feller semigroup on $\mathcal{C}_0(\mathsf{E})$ with generator $(\mathcal{L}, D(\mathcal{L}))$ and X be a corresponding Feller process. Suppose that $r \in \mathcal{C}_b(\mathsf{E})$ with $r \geq 0$ and a multiplicative functional M is defined by

$$M(t) = e^{-\int_0^t r(X(s))\mathrm{d}s}.$$

At the same time, a semigroup entitles to a definition below.

$$\mathcal{P}^M(t)u(x) := \boldsymbol{E}^x\big[M(t)u(X(t))\big].$$

The next lemma shows when this semigroup becomes a Feller semigroup.

Lemma 2.23. *If* $\{\mathcal{P}_t\}_{t\geq 0}$ *is a Feller semigroup,* $r \in \mathcal{C}_b(\mathsf{E})$ *and* $r \geq 0$*. Then,* $\{\mathcal{P}_t^M\}_{t\geq 0}$ *is also a Feller semigroup with generator* $(\mathcal{L}^M, D(\mathcal{L}))$*, where*

$$\mathcal{L}^M u(x) := \mathcal{L}u(x) - r(x)u(x).$$

Remark 2.24. *This sets the scene for the type of perturbations. However,* $\{\mathcal{P}_t^M\}_{t\geq 0}$ *being a Feller semigroup does not require a continuously bounded* r*. In fact, [Böttcher et al., 2013, Section 4.4] introduced the case when* r *is not necessarily continuous but satisfying the abstract Kato condition, i.e.,*

$$\lim_{t\to 0} \sup_{x\in\mathsf{E}} \int_0^t \mathcal{P}_s|r|(x)\mathrm{d}s = 0.$$

Now let us look at an example under (3) as in Example 2.21. This example describes a standard Brownian motion whose killing time is the time when it reaches 0. Let $\{X(t)\}_{t\geq 0} = \{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Define

$$\begin{aligned} \zeta = \tau_0 &:= \inf\{t > 0; X(t) \notin (0,\infty)\}, \\ M^\zeta &= \{\mathbf{1}_{\tau_0 < t}\}_{t\geq 0}. \end{aligned}$$

Further define

$$\mathcal{P}_t^\zeta u(x) := \boldsymbol{E}^x\big[M(t)u(X(t))\big] = \boldsymbol{E}^x\big[u(X(t))\mathbf{1}_{\tau_0<t}\big].$$

$\{\mathcal{P}_t^\zeta\}_{t\geq 0}$ is a Feller semigroup on $\mathcal{C}_0((0,\infty))$ whose core operator is given by

$$\begin{aligned} D(\mathcal{G}_{kill}^{BC}) &:= \{u \in \mathcal{C}_0^2((0,\infty));\ D_{xx}u \in \mathcal{C}_0((0,\infty))\}, \\ \mathcal{G}_{kill}^{BC}u(x) &:= \frac{1}{2}D_{xx}u(x). \end{aligned}$$

Remark 2.25. *Note that optimal stopping problems with operator* $D(\mathcal{G}_{kill}^{BC})$ *cannot be defined in Chapter 3. Nevertheless, by incoporating a multiplicative functional, we are able to fix this issue. Chapter 5 will emphasise this in more details.*

Chapter 3

Viscosity Solutions for Optimal Stopping Problems for Feller Processes

This chapter studies an optimal stopping problem when the state process is governed by a general Feller process. In particular, we examine viscosity properties of the associated value function with no a priori assumption made on the stochastic differential equation satisfied by the state process. Our approach relies on properties of the Feller semigroup. We present conditions on the process under which the value function is the unique viscosity solution to an Hamilton-Jacobi-Bellman (HJB) equation associated with a particular operator. More specifically, assuming that controlled process is a Feller process, we prove uniqueness of the viscosity solution which was conjectured in Palczewski and Stettner [2014]. We then apply our results to study viscosity property of optimal stopping problems for some particular Feller processes, namely diffusion processes with piecewise coefficients and semi-Markov processes. Finally, we obtain explicit value functions for optimal stopping of straddle options, when the state process is a reflected Brownian motion, Brownian motion with jump at boundary and regime switching Feller diffusion (see Section 3.7).

The rest of this chapter is organized as follows. Section 3.1 introduces terminologies used throughout this chapter and then formulate the optimal stopping problem. Section 3.2 investigates the link between the value function of an optimal stopping problem and the viscosity solution to an HJB equation when the state. Section 3.3 studies uniqueness of the viscosity solution and its link to the value function under the assumption that the state space is compact. The proof relies on the comparison theorem (Theorem 3.15.) Section 3.4 examines the extension of the uniqueness to the case of non compact state space. Section 3.5 study the structure of the viscosity solution and its link to the martingale approach. In Section 3.6, we apply our result to study viscosity property of value functions of optimal stopping problems for some processes satisfying our key assumptions. Section 3.7 is devoted to the derivation of explicit value function for optimal stopping of a straddle option.

3.1 Problem Formulation

In this chapter, we study an optimal stopping problems for a normal Markov process $X := (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ on the state space $(E, \mathcal{E})$, where $(\Omega, \mathcal{F})$ is a measurable space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous and completed filtration, $\{X(t)\}_{t\geq 0}$ is a *càdlàg* stochastic process, $\{\theta_t\}_{t\geq 0}$ is the shift operator and $\boldsymbol{P}^x$ denotes the probability measure on $(\Omega, \mathcal{F})$ for $x \in \mathsf{E}$. Let $\mathcal{T}$ be the family of all $\mathcal{F}_t$-stopping times. Let f and g be two real-valued Borel measurable functions on E. Define the objective function $J_x(\tau)$ by

$$J_x(\tau) := \boldsymbol{E}^x\Big[\int_0^\tau e^{-as} f(X(s))\,\mathrm{d}s + e^{-a\tau} g(X(\tau))\Big] \text{ for } x \in \mathsf{E} \text{ and } \tau \in \mathcal{T}, \quad (3.1.1)$$

where f is a running benefit function, g is a terminal reward function and $a > 0$ is a constant discount factor.

We consider the following optimal stopping problem: find $\tau^* \in \mathcal{T}$ such that

$$V(x) := \sup_{\tau \in \mathcal{T}} J_x(\tau) = J_x(\tau^*), \quad (3.1.2)$$

for each $x \in \mathsf{E}$. Our main goal is to study properties of the value function V.

For this purpose, we first give the definition of viscosity solution:

Definition 3.1. *(Viscosity Solution) Given an operator with domain* $(\mathcal{A}, D(\mathcal{A}))$*, a function* $w \in USC(\mathsf{E})$ *(respectively,* $w \in LSC(\mathsf{E})$*) is a viscosity subsolution (respectively, supersolution) associated with* $(\mathcal{A}, D(\mathcal{A}))$ *to (3.1.23) if for all* $\phi \in D(\mathcal{A})$ *such that* $\phi - w$ *has a global minimum (respectively, maximum) at* $x_0 \in \mathsf{E}$ *with* $\phi(x_0) = w(x_0)$*,*

$$\min\left(a\phi(x_0) - \mathcal{A}\phi(x_0) - f(x_0), \phi(x_0) - g(x_0)\right) \leq (\geq) 0. \quad (3.1.3)$$

Furthermore, $w \in \mathcal{C}(\mathsf{E})$ *is a viscosity solution associated with* $(\mathcal{A}, D(\mathcal{A}))$ *to (3.1.23) if it is both a viscosity supersolution and a viscosity subsolution.*

Next, let us introduce the notion of a-generator:

Definition 3.2. *(a-generator) Let* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ *be a Markov process on the state space* $(\mathsf{E}, \mathcal{E})$*. Set* $a > 0$*. An operator* $(\mathcal{A}, D(\mathcal{A}))$ *is called an a-supergenerator (respectively, a-subgenerator, a-generator) of* X*, if for any* $w \in$

$D(\mathcal{A})$, the process $\{S_w(t)\}_{t\geq 0}$ defined by

$$S_w(t) := w(X(0)) - e^{-at}w(X(t)) - \int_0^t e^{-as}(aw - \mathcal{A}w)(X(s))\mathrm{d}s, \tag{3.1.4}$$

is a $(\mathcal{F}_t, \boldsymbol{P}^x)$ uniformly integrable supermartingale (respectively, submartingale, martingale) for all $x \in \mathsf{E}$.

The following assumptions holds throughout this chapter.

Assumption 1.

1. *E is a locally compact, separable metric space with metric ρ.*
2. *$X := (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ is a Feller process with the state space $(\mathsf{E}, \mathcal{E})$, which has a Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$, whose generator is $(\mathcal{L}, D(\mathcal{L}))$ with a core $(\mathcal{G}, D(\mathcal{G}))$.*
3. *$a > 0$ and $f, g \in \mathcal{C}_b(\mathsf{E})$.*

Assumption 1 does not make any a priori assumption on the partial differential equation satisfied by the generator of the Feller process. We first recall a result on the continuity of the value function V given by (3.1.2). The proof of the continuity is based on the penalty method which consists in finding a sequence $\{v_\lambda\}_{\lambda>0}$ in $\mathcal{C}_0(\mathsf{E})$ that converges uniformly to the value function V. More precisely, the penalty function v_λ is defined as the solution to the following equation

$$av_\lambda - \mathcal{L}v_\lambda - f = \lambda(g - v_\lambda)^+, \tag{3.1.5}$$

where $\lambda > 0$. The next results which are similar to [Robin, 1978, Theorem I.2.1 and Theorem I.3.1] provide the continuity of the value function.

Theorem 3.3. *Under Assumption 1,*

1. *Equation (3.1.5) admits a unique solution $v_\lambda \in D(\mathcal{L})$ for each $\lambda > 0$.*
2. *The value function V defined by (3.1.2) is in $\mathcal{C}_0(\mathsf{E})$. In addition, $\{v_\lambda\}_{\lambda>0}$ defined by (3.1.5) converges uniformly to V from below as $\lambda \to \infty$.*

Proof.
(1) Define the penalty function v_λ as the solution to

$$v_\lambda = \mathcal{R}_a(f + \lambda(g - v_\lambda)^+). \tag{3.1.6}$$

We start by showing that (3.1.6) has a unique solution in $\mathcal{C}_0(\mathsf{E})$ in the following lemma.

Lemma 3.4. *Suppose that Assumption 1 holds. For any $\lambda > 0$,* (3.1.6) *admits a unique solution $v_\lambda \in \mathcal{C}_0(\mathsf{E})$. Additionally, the solution to* (3.1.6) *is equivalent to the solution to the following equation*

$$v_\lambda = \mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda). \tag{3.1.7}$$

(We say that the equivalence between two equations means that, the solution to the one is also a solution to the other, vice versa.)

Proof. We first show that the solution to (3.1.6) is equivalent to the solution to (3.1.7). Let v_λ be the solution to (3.1.6) in $\mathcal{C}_0(\mathsf{E})$. Using the resolvent identity equation (2.2.2), we obtain

$$\mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+) - \mathcal{R}_a(f + \lambda(g - v_\lambda)^+) = -\lambda\mathcal{R}_{a+\lambda}\mathcal{R}_a(f + \lambda(g - v_\lambda)^+). \tag{3.1.8}$$

Combining (3.1.6) and (3.1.8), we have

$$\mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+) - v_\lambda = -\lambda\mathcal{R}_{a+\lambda}v_\lambda.$$

Therefore, v_λ is also a solution to (3.1.7).
Now, let v_λ be a solution to (3.1.7). Using once more (2.2.2), we have

$$\begin{aligned}&\mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda) - \mathcal{R}_a(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda)\\ =& -\lambda\mathcal{R}_a\mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda).\end{aligned} \tag{3.1.9}$$

Combining (3.1.7) and (3.1.9) yields

$$v_\lambda - \mathcal{R}_a(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda) = -\lambda\mathcal{R}_a v_\lambda.$$

Hence, v_λ is also a solution to (3.1.6).

In order to complete the proof, it is enough to show that (3.1.7) has the unique solution. Define a new operator $\mathcal{Z}$ as follows:

$$\mathcal{Z}w := \mathcal{R}_{a+\lambda}(f + \lambda(g - w)^+ + \lambda w).$$

We have that $f, g \in \mathcal{C}_0(\mathsf{E})$ and the resolvent operator maps from $\mathcal{C}_0(\mathsf{E})$ to $\mathcal{C}_0(\mathsf{E})$. Let $w \in \mathcal{C}_0(\mathsf{E})$, then $\mathcal{Z}w$ is also in $\mathcal{C}_0(\mathsf{E})$. Furthermore, let $w_1, w_2 \in \mathcal{C}_0(\mathsf{E})$. Using the linearity of the resolvent and the fact that $(g - w_i)^+ + w_i = \max(g, w_i)$ for

$i = 1, 2$, we have

$$\begin{aligned}
\|\mathcal{Z}w_1 - \mathcal{Z}w_2\|_\infty &= \|\mathcal{R}_{a+\lambda}(f + \lambda(g - w_1)^+ + \lambda w_1) - \mathcal{R}_{a+\lambda}(f + \lambda(g - w_2)^+ + \lambda w_2)\|_\infty \\
&= \|\lambda \mathcal{R}_{a+\lambda}(\max(g, w_1) - \max(g, w_2))\|_\infty \\
&\leq \frac{\lambda}{a+\lambda}\|\max(g, w_1) - \max(g, w_2))\|_\infty \\
&\leq \frac{\lambda}{a+\lambda}\|w_1 - w_2\|_\infty,
\end{aligned}$$

where the inequality comes from (2.2.4). Hence, $\mathcal{Z}$ is a contraction mapping from $\mathcal{C}_0(\mathsf{E})$ to $\mathcal{C}_0(\mathsf{E})$. By Banach fixed point theorem, the equation $w = \mathcal{Z}w$ (which is the same as (3.1.7)) has a unique solution $w \in \mathcal{C}_0(\mathsf{E})$, that we denote by v_λ. ■

Recall that the operator $(\lambda - \mathcal{L})$ is a bijection of $D(\mathcal{L})$ to $\mathcal{C}_0(\mathsf{E})$ and its inverse is the resolvent $\mathcal{R}_\lambda$ (see Corollary 2.8). The solution to (3.1.6) is equivalent to the solution to (3.1.5). It remains to show that $v_\lambda \in D(\mathcal{L})$. We have shown that (3.1.6) has the unique solution v_λ in $\mathcal{C}_0(\mathsf{E})$ such that $f + \lambda(g - w)^+ \in \mathcal{C}_0(\mathsf{E})$. Therefore, (1) in Theorem 3.3 is proved.

(2) Let v_λ be the unique solution in $D(\mathcal{L})$ to (3.1.5) for $\lambda > 0$. We prove that the sequence of penalty functions $\{v_\lambda\}_{\lambda>0}$ converges uniformly to the value function V in $\mathcal{C}_0(\mathsf{E})$ as $\lambda \to \infty$. We need the following two preliminary lemmas.

Lemma 3.5. *Suppose that Assumption 1 holds. Let $\{g_n\}_{n\mathbb{N}^+}$ be a sequence in $\mathcal{C}_0(\mathsf{E})$ such that*

$$\|g - g_n\|_\infty \leq \frac{1}{n}. \tag{3.1.10}$$

Define a sequence of the corresponding value functions $\{V_n\}_{n\in\mathbb{N}^+}$ by

$$V_n(x) := \sup_{\tau\in\mathcal{T}} J_x^{(n)}(\tau) \text{ for } x \in \mathsf{E} \text{ and } n \in \mathbb{N}^+, \tag{3.1.11}$$

where $J_x^{(n)}(\tau) = \boldsymbol{E}^x\left[\int_0^\tau e^{-as} f(X(s))ds + e^{-a\tau} g_n(X(\tau))\right]$. Then, V_n converges to V defined by (3.1.2) uniformly as $n \to \infty$.

Proof. Let $x \in \mathsf{E}$, $n \in \mathbb{N}^+$ and $\varepsilon > 0$. Define a ε-optimal stopping time τ_ε^* such that

$$V(x) - \varepsilon \leq J_x(\tau_\varepsilon^*). \tag{3.1.12}$$

Therefore, we have

$$\begin{aligned} V(x) &\leq \boldsymbol{E}^x\left[\int_0^{\tau_\varepsilon^*} e^{-as} f(X(s))ds + e^{-a\tau_\varepsilon^*} g(X(\tau_\varepsilon^*))\right] + \varepsilon \\ &\leq \boldsymbol{E}^x\left[\int_0^{\tau_\varepsilon^*} e^{-as} f(X(s))ds + e^{-a\tau_\varepsilon^*} (g_n(X(\tau_\varepsilon^*)) + \|g - g_n\|_\infty)\right] + \varepsilon \\ &\leq J_x^{(n)}(\tau_\varepsilon^*) + \frac{1}{n} + \varepsilon \leq V_n(x) + \frac{1}{n} + \varepsilon. \end{aligned}$$

Since ε is an arbitrary positive constant, $V(x) - V_n(x) \leq \frac{1}{n}$. On the other hand, we can find a stopping time $\tau_\varepsilon^{*(n)}$ for V_n such that $V_n(x) - \varepsilon \leq J_x^{(n)}(\tau_\varepsilon^{*(n)})$. We can also obtain $V_n(x) - V(x) \leq \frac{1}{n}$ similarly. Therefore, we have

$$\|V - V_n\|_\infty \leq \frac{1}{n}. \tag{3.1.13}$$

Then, the proof is completed. ■

Lemma 3.6. *Suppose that Assumption 1 holds. Let v_λ be the solution to (3.1.6) for each $\lambda > 0$. Then, v_λ satisfies*

$$v_\lambda(x) = \sup_\tau \boldsymbol{E}^x\left[\int_0^\tau e^{-as} f(X(s))ds + e^{-a\tau}(g - (g - v_\lambda)^+)(X(\tau))\right]. \tag{3.1.14}$$

Additionally, $V \geq v_\lambda$.

Proof. Let $x \in \mathsf{E}$, $\lambda > 0$ and τ be a $\mathcal{F}_t$-stopping time. We know from **(1)** in Theorem 3.3 that $v_\lambda \in D(\mathcal{L})$. Then, using Dynkin's formula and optional stopping theorem,

$$\begin{aligned} v_\lambda(x) &= \boldsymbol{E}^x\left[\int_0^\tau e^{-as}(f(X(s)) + \lambda(g - v_\lambda)^+(X(s)))ds + e^{-a\tau} v_\lambda(X(\tau))\right]. \\ &\geq \boldsymbol{E}^x\left[\int_0^\tau e^{-as} f(X(s))ds + e^{-a\tau}(g - (g - v_\lambda)^+)(X(\tau))\right]. \end{aligned} \tag{3.1.15}$$

Taking the supremum on both sides, we obtain

$$v_\lambda(x) \geq \sup_\tau \boldsymbol{E}^x\left[\int_0^\tau e^{-as} f(X(s))ds + e^{-a\tau}(g - (g - v_\lambda)^+)(X(\tau))\right]. \tag{3.1.16}$$

In order to prove the equality, define the stopping time σ^* by $\sigma^* := \inf\{s \geq 0;\ v_\lambda(X(s)) \leq g(X(s))\}$. Since $\{X\}_{t\geq 0}$ is right continuous and v_λ and g are continuous, we have $v_\lambda(X(\sigma^*)) \leq g(X(\sigma^*))$. Using the preceding and (3.1.15), we have

$$\begin{aligned}
v_\lambda(x) &= \boldsymbol{E}^x\Big[\int_0^{\sigma^*} e^{-as}(f(X(s)) + \lambda(g - v_\lambda)^+(X(s)))ds + e^{-a\sigma^*}v_\lambda(X(\sigma^*))\Big] \\
&= \boldsymbol{E}^x\Big[\int_0^{\sigma^*} e^{-as}f(X(s)ds + e^{-a\sigma^*}(g - (g - v_\lambda)^+)(X(\sigma^*))\Big].
\end{aligned}$$

Hence, (3.1.14) is proved. Furthermore, since $g - (g - v_\lambda)^+ \leq g$, (3.1.14) implies $V \geq v_\lambda$ for all $\lambda > 0$. The proof is completed. ■

Lemma 3.7. *Suppose that Assumption 1 holds. Assume in addition that $g \in D(\mathcal{L})$. Let v_λ be the solution to (3.1.5) for $\lambda > 0$. v_λ converges to V uniformly as $\lambda \to \infty$ and hence $V \in \mathcal{C}_0(\mathsf{E})$.*

Proof. Let $x \in \mathsf{E}$ and $\lambda > 0$. Using (3.1.14), we get

$$\begin{aligned}
v_\lambda(x) &= \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}f(X(s))ds + e^{-a\tau}(g - (g - v_\lambda)^+)(X(\tau))\Big] \\
&\geq \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}f(X(s))ds + e^{-a\tau}g(X(\tau))\Big] - \sup_\tau \boldsymbol{E}^x\Big[e^{-a\tau}(g - v_\lambda)^+(X(\tau))\Big] \\
&\geq V(x) - \sup_\tau \boldsymbol{E}^x\left[(g - v_\lambda)^+(X(\tau))\right] \\
&\geq V(x) - \|(g - v_\lambda)^+\|_\infty.
\end{aligned}$$

Additionally, we have from Lemma 3.6 that $V \geq v_\lambda$ for all $\lambda > 0$. Then,

$$\|V - v_\lambda\|_\infty \leq \|(g - v_\lambda)^+\|_\infty. \tag{3.1.17}$$

Furthermore, since $v_\lambda \in D(\mathcal{L})$ by **(1)** in Theorem 3.3, $g - v_\lambda \in D(\mathcal{L})$ and thus

$$g - v_\lambda = \mathcal{R}_a((a - \mathcal{L})(g - v_\lambda)).$$

Hence, using (3.1.5) and similar argument as in (3.1.8), we get

$$\begin{aligned}
g - v_\lambda &= \mathcal{R}_a((a - \mathcal{L})g - (a - \mathcal{L})v_\lambda) \\
&= \mathcal{R}_a((a - \mathcal{L})g - f - \lambda(g - v_\lambda)^+) \\
&= \mathcal{R}_{a+\lambda}((a - \mathcal{L})g - f - \lambda(g - v_\lambda)^+ + \lambda(g - v_\lambda)) \\
&\leq \mathcal{R}_{a+\lambda}((a - \mathcal{L})g - f - \lambda(g - v_\lambda)^+ + \lambda(g - v_\lambda)^+) \qquad (3.1.18) \\
&\leq \frac{\|(a - \mathcal{L})g - f\|_\infty}{a + \lambda}.
\end{aligned}$$

Thus, it follows from (3.1.17) that

$$\|V - v_\lambda\|_\infty \le \|(g - v_\lambda)^+\|_\infty \le \frac{\|(a - \mathcal{L})g - f\|_\infty}{a + \lambda} \le \frac{M}{a + \lambda}, \tag{3.1.19}$$

where $M > 0$ is a constant. Hence, v_λ converges to V uniformly as $\lambda \to \infty$. The proof of (1) in Theorem 3.3 is completed. ■

It remains to show that the conclusion of Lemma 3.7 is also true for any $g \in \mathcal{C}_0(\mathsf{E})$. Let $g \in \mathcal{C}_0(\mathsf{E})$. Since $D(\mathcal{L})$ is dense in $\mathcal{C}_0(\mathsf{E})$ (see Theorem 2.7), then there exists a sequence $\{g_n\}_{n\in\mathbb{N}^+}$ in $D(\mathcal{L})$ uniformly converging to g as $n \to \infty$ such that $\|g_n - g\|_\infty \le 1/n$ for any $n \in \mathbb{N}^+$. Using Lemma 3.7, the sequence of the value functions $\{V_n\}_{n\in\mathbb{N}^+}$ defined by (3.1.11) is in $\mathcal{C}_0(\mathsf{E})$. Using Lemma 3.5, $\{V_n\}_{n\in\mathbb{N}^+}$ converges to V uniformly as $n \to \infty$. Therefore, $V \in \mathcal{C}_0(\mathsf{E})$.

Moreover, let $v_{n,\lambda}$ be the solution to (3.1.5) after replacing g by g_n for each $n \in \mathbb{N}^+$. Let us prove that $v_{n,\lambda}$ converges to v_λ uniformly as $n \to \infty$. Using (3.1.7),

$$\begin{aligned}
v_{n,\lambda} - v_\lambda &= \mathcal{R}_{a+\lambda}(f + \lambda(g_n - v_{n,\lambda})^+ + \lambda v_{n,\lambda}) - \mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda) \\
&= \lambda \mathcal{R}_{a+\lambda}(\max(g_n, v_{n,\lambda}) - \max(g, v_\lambda)) \\
&\le \lambda \mathcal{R}_{a+\lambda}(\max(g_n - g, v_{n,\lambda} - v_\lambda)) \\
&\le \frac{\lambda}{a+\lambda} \|\max(g_n - g, v_{n,\lambda} - v_\lambda)\|_\infty \\
&\le \frac{\lambda}{a+\lambda} \max(\|g - g_n\|_\infty, \|v_{n,\lambda} - v_\lambda\|_\infty).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
v_\lambda - v_{n,\lambda} &= \mathcal{R}_{a+\lambda}(f + \lambda(g - v_\lambda)^+ + \lambda v_\lambda) - \mathcal{R}_{a+\lambda}(f + \lambda(g_n - v_{n,\lambda})^+ + \lambda v_{n,\lambda}) \\
&\le \frac{\lambda}{a+\lambda} \max(\|g - g_n\|_\infty, \|v_{n,\lambda} - v_\lambda\|_\infty).
\end{aligned}$$

Therefore, we obtain

$$\|v_\lambda - v_{n,\lambda}\|_\infty \le \frac{\lambda}{a+\lambda} \max(\|g - g_n\|_\infty, \|v_{n,\lambda} - v_\lambda\|_\infty).$$

Since $\frac{\lambda}{a+\lambda} < 1$, it follows that

$$\|v_{n,\lambda} - v_\lambda\|_\infty \le \|g_n - g\|_\infty \le \frac{1}{n}. \tag{3.1.20}$$

Then, $v_{n,\lambda}$ converges to v_λ uniformly as $n \to \infty$.

Now, let $\varepsilon > 0$ and choose an integer $n_0 > \frac{4}{\varepsilon}$. Hence, combing (3.1.13), (3.1.19) and (3.1.20) yield

$$\begin{aligned}\|V - v_\lambda\|_\infty &\le \|V - V_{n_0}\|_\infty + \|v_{n_0,\lambda} - v_\lambda\|_\infty + \|V - v_{n_0,\lambda}\|_\infty \\ &\le \frac{1}{n_0} + \frac{1}{n_0} + \frac{M_{n_0}}{a+\lambda} \le \frac{\varepsilon}{2} + \frac{M_{n_0}}{a+\lambda},\end{aligned} \tag{3.1.21}$$

where $M_{n_0} = \|(a-\mathcal{L})g_{n_0} - f\|_\infty$. Therefore, $\|V - v_\lambda\|_\infty \le \varepsilon$ for any $\lambda > \frac{2M_{n_0}}{\varepsilon}$. Thus, the proof is completed. ■

For more information on the continuity of the value function and its extensions; readers are referred to Palczewski and Stettner [2010, 2011, 2014]; Robin [1978]; Stettner [2011]; Stettner and Zabczyk [1983]). The optimal stopping time for the above optimal stopping problem can be obtained using [Robin, 1978, Theorem I.3.3] as follows.

Theorem 3.8. *Under Assumption 1, the optimal stopping time is*

$$\tau^* := \inf\{t \ge 0; V(X(t)) = g(X(t))\}. \tag{3.1.22}$$

Let $(\mathcal{A}, D(\mathcal{A}))$ denotes an operator with its domain. Recall that, we wish to study the link between the value function V defined by (3.1.2) and the unique viscosity solution associated with $(\mathcal{A}, D(\mathcal{A}))$ to the corresponding Hamilton-Jacob-Bellman (HJB) equation

$$\min\left(aw - \mathcal{A}w - f, w - g\right) = 0. \tag{3.1.23}$$

3.2 Existence of viscosity solution

In this, we show that the value function defined by (3.1.2) can be described as a viscosity solution associated with the generator $(\mathcal{L}, D(\mathcal{L}))$ of the Feller process or its core $(\mathcal{G}, D(\mathcal{G}))$.We prove that the value function defined by (3.1.2) is a viscosity supersolution (*respectively*, subsolution, solution) associated with an extended generator of the Feller process.

Theorem 3.9. *Suppose Assumption 1 holds. Suppose* $(\mathcal{A}, D(\mathcal{A}))$ *is an a-supergenerator (respectively, a-subgenerator, a-generator) of* X *and* $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{C}(\mathsf{E}) \to \mathcal{C}(\mathsf{E})$. *Then the value function* V *defined by* (3.1.2) *is a viscosity supersolution (respectively, subsolution, solution) associated with* $(\mathcal{A}, D(\mathcal{A}))$ *to*

$$\min(aw - \mathcal{A}w - f, w - g) = 0. \tag{3.2.1}$$

Proof. The method used to show the existence is based on the probabilistic description of the extended generator of the Feller process $\{X(t)\}_{t\ge 0}$. See

Section 3.2.1 for a detailed proof. ■

Remark 3.10. *As we will see later, enlarging the domain $D(\mathcal{A})$ has the advantage that it allows to exclude a function which is not a viscosity solution. Hence in this thesis, we use the solution of the martingale problem to define the extended generator instead of the infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$. The former enables us to provide more choices on the test function in $D(\mathcal{A})$. For example, $D(\mathcal{A})$ can be chosen to be $\mathcal{C}_b^2(\mathsf{E})$ or could even include an unbounded function space; see for instance Section 3.6.1.1.*

One can also show as in [Costantini and Kurtz, 2015, Lemma 2.9] that if the process $\{S_w^{(0)}(t)\}_{t\geq 0}$ defined by

$$S_w^{(0)}(t) := w(X_0) - w(X_t) + \int_0^t \mathcal{A}w(X(s))\mathrm{d}s$$

is a $(\mathcal{F}_t, \boldsymbol{P}^x)$-martingale for any $x \in \mathsf{E}$ and $w \in D(\mathcal{A})$, then $(\mathcal{A}, D(\mathcal{A}))$ is an a-generator for $a > 0$ when $\mathcal{A} : D(\mathcal{A}) \subseteq B(\mathsf{E}) \to B(\mathsf{E})$. Therefore, by Dynkin's formula, the infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$ of the Feller process or its core $(\mathcal{G}, D(\mathcal{G}))$ is an a-generator for all $a > 0$. In this case, Theorem 3.9 implies the following corollary

Corollary 3.11. *Under Assumption 1, the value function V defined by (3.1.2) is a viscosity solution associated with the infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$ to*

$$\min\left(aw - \mathcal{L}w - f, w - g\right) = 0. \tag{3.2.2}$$

The proofs of the above results are given by the following section.

3.2.1 Proof of Theorem 3.9

This section is devoted to the proof of Theorem 3.9. The proof is standard with a modification due to the presence of the absorbing state. The proof will be given for two classes of the initial state $x \in E$: the absorbing and the non-absorbing states.

We say that $x \in \mathsf{E}$ is an absorbing state if and only if $X_t = x$ for all $t \in [0, \infty)$ almost surely under $\boldsymbol{P}^x$. Let τ_δ be an $\mathcal{F}_t$-stopping time defined by

$$\tau_\delta := \inf\{s \geq 0; X(s) \notin \bar{B}(X(0), \delta)\}, \tag{3.2.3}$$

where $\delta > 0$ and $B(x, \delta) := \{y \in \mathsf{E};\ \rho(x, y) < \delta\}$. The following lemma that can be found in [Kallenberg, 2006, Lemma 17.22] provides information on the stopping time τ_δ when the initial state x is absorbing or not.

Lemma 3.12. *Let X be a Feller process.*

1. *Assume $x \in \mathsf{E}$ is not absorbing. Then $\boldsymbol{E}^x[\tau_\delta] < \infty$ for all sufficiently small $\delta > 0$,*

2. *$x \in \mathsf{E}$ is absorbing if and only if $\boldsymbol{P}(\tau_\delta = \infty) = 1$ for all $\delta > 0$.*

The subsequent lemmas are needed in the proof of the existence of the viscosity solution for absorbing initial state process. Their proofs are standard. However for the sake of completeness, we provide details.

Lemma 3.13. *Suppose Assumption 1 holds. Suppose in addition that the initial state $x \in \mathsf{E}$ is absorbing. Then the value function satisfies*

$$V(x) = \max\Big(\frac{f(x)}{a}, g(x)\Big). \tag{3.2.4}$$

Proof. Since the initial state $x \in \mathsf{E}$ of the Feller process X is absorbing, we have $X_t = x$ for all $t \in [0, \infty)$ $\boldsymbol{P}^x$-a.s. For $x \in \mathsf{E}$,

$$\begin{aligned} V(x) =& \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as} f(x) \mathrm{d}s + e^{-a\tau} g(x)\Big] \\ =& \sup_\tau \boldsymbol{E}^x\Big[\frac{f(x)(1 - e^{-a\tau})}{a} + e^{-a\tau} g(x)\Big] \\ =& \sup_\tau \boldsymbol{E}^x\Big[\frac{f(x)}{a} + e^{-a\tau}\Big(g(x) - \frac{f(x)}{a}\Big)\Big]. \end{aligned}$$

If $g(x) > \frac{f(x)}{a}$, then $V(x) \le g(x)$ and the equality is attained on the set $\{\tau = 0\}$, that is, $V(x) = g(x)$, otherwise, $V(x) \le \frac{f(x)}{a}$ and the equality is attained on the set $\{\tau = \infty\}$, that is, $V(x) = \frac{f(x)}{a}$. ∎

Lemma 3.14. *Suppose Assumption 1 holds. For $x \in \mathsf{E}$ and $\delta > 0$,*

$$V(x) \ge \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau_\delta} V(X(\tau_\delta))\Big]. \tag{3.2.5}$$

Suppose in addition that $V(x) > g(x)$. Then there exists a constant $\Delta > 0$ such that for any $\delta \le \Delta$, we have

$$V(x) = \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau_\delta} V(X(\tau_\delta))\Big]. \tag{3.2.6}$$

Proof. Let $Z(t) := \int_0^t e^{-as} f(X(s)) \mathrm{d}s + e^{-at} V(X(t))$ for $t \ge 0$. Then, using Snell envelope (see for example [Peskir and Shiryaev, 2006a, Theorem 2.4]), the process $\{Z_t\}_{t\geq 0}$ is a supermartingale and $\{Z_t\}_{t\wedge\tau^*}$ is a martingale, where τ^* is defined by (3.1.22). Therefore, (3.2.5) and (3.2.6) follows. In particular, (3.2.6) follows from the fact that E is a separable metric space and V and g are continuous. ■

We now turn to the proof of Theorem 3.9.

Proof of Theorem 3.9.

(1) Viscosity Supersolution: Suppose $(\mathcal{A}, D(\mathcal{A}))$ is an a-supergenerator of the Markov process X. Suppose $x \in \mathsf{E}$ and $\phi \in D(\mathcal{A})$ such that $\phi(x) = V(x)$ and $\phi - V$ has a global maximum at $x \in \mathsf{E}$. We wish to prove that

$$\min(a\phi(x) - \mathcal{A}\phi(x) - f(x), \phi(x) - g(x)) \geq 0.$$

Since $\phi(x) = V(x) \geq g(x)$, it is sufficient to prove that

$$a\phi(x) - \mathcal{A}\phi(x) - f(x) \geq 0. \tag{3.2.7}$$

Case 1. Assume that $x \in \mathsf{E}$ is an absorbing initial state that is $X_t = x$ for all $t \in [0, \infty)$ $\boldsymbol{P}^x$-a.s. and define the process $\{S_\phi(t)\}_{t\geq 0}$ by

$$S_\phi(t) := \phi(X_0) - e^{-at}\phi(X_t) - \int_0^t e^{-as}(a\phi - \mathcal{A}\phi)(X(s))\mathrm{d}s.$$

Since $x \in \mathsf{E}$ is absorbing, $S_\phi(t) = \int_0^t e^{-as}\mathcal{A}\phi(x)\mathrm{d}s$ for all $t \in [0, \infty)$ $\boldsymbol{P}^x$-a.s. Since $(\mathcal{A}, D(\mathcal{A}))$ is an a-supergenerator, it follows that $\{S_\phi(t)\}_{t\geq 0}$ is a $(\mathcal{F}_t, \boldsymbol{P}^x)$ uniformly integrable supermartingale, and therefore $\mathcal{A}\phi(x) \leq 0$. In addition, using Lemma 3.13, we have $\phi(x) = V(x) = \max(f(x)/a, g(x))$. The latter combines with the fact that $\mathcal{A}\phi(x) \leq 0$ yields (3.2.7).

Case 2. Assume that $x \in \mathsf{E}$ is not an absorbing initial value. It follows from Lemma 3.12 that $\boldsymbol{E}^x[\tau_\delta] < \infty$ for all small enough $\delta > 0$. Since $\phi \in D(\mathcal{A})$ and $\phi(y) - V(y) \leq 0$ for any $y \in \mathsf{E}$, (3.2.5) implies that

$$\begin{aligned} V(x) \geq& \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}f(X(s))\mathrm{d}s + e^{-a\tau_\delta}V(X(\tau_\delta))\Big] \\ \geq& \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}f(X(s))\mathrm{d}s + e^{-a\tau_\delta}\phi(X(\tau_\delta))\Big] \\ \geq& \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}(f(X(s)) + \mathcal{A}\phi(X(s)) - a\phi(X(s)))\mathrm{d}s\Big] + \phi(x), \end{aligned} \tag{3.2.8}$$

where the last inequality follows from the optional sampling theorem since $(\mathcal{A}, D(\mathcal{A}))$ is an a-supergenerator. Since $V(x) = \phi(x)$, dividing both sides of (3.2.8) by

$\boldsymbol{E}^x\big[\tau_\delta\big]$, we obtain

$$
\begin{aligned}
0 \geq & \frac{\boldsymbol{E}^x\big[\int_0^{\tau_\delta} e^{-as}(f(X(s))+\mathcal{A}\phi(X(s))-a\phi(X(s)))\mathrm{d}s\big]}{\boldsymbol{E}^x\big[\tau_\delta\big]} \\
\geq & \frac{\boldsymbol{E}^x\big[\int_0^{\tau_\delta} e^{-as}C_-(x,\delta)\mathrm{d}s\big]}{\boldsymbol{E}^x\big[\tau_\delta\big]} \\
= & \frac{1-\boldsymbol{E}^x[e^{-a\tau_\delta}]}{\boldsymbol{E}^x\big[\tau_\delta\big]}C_-(x,\delta),
\end{aligned} \tag{3.2.9}
$$

where $C_-(x,\delta)=\inf_{y\in B(x,\delta)}(f(y)+\mathcal{A}\phi(y)-a\phi(y))$. Since $\boldsymbol{E}^x\big[\tau_\delta\big]$ is bounded such that $\frac{1-\boldsymbol{E}^x[e^{-a\tau_\delta}]}{\boldsymbol{E}^x\big[\tau_\delta\big]}>0$, (3.2.9) yields $C_-(x,\delta)\leq 0$ for all $\delta>0$. Since $f,\mathcal{A}\phi$ and ϕ are in $\mathsf{C}(\mathsf{E})$, $C_-(x,\delta)$ converges pointwise to $f(x)+\mathcal{A}\phi(x)-a\phi(x)$ as $\delta\to 0$. Hence, (3.2.7) is proved.

(2) Viscosity Subsolution: Assume that $(\mathcal{A},D(\mathcal{A}))$ is an a-subgenerator of the process X. Choose $\psi\in D(\mathcal{A})$, such that $\psi(x)=V(x)$ and $\psi-V$ has a global minimum at x. If $V(x)=g(x)$, we find a viscosity subsolution by setting $\psi(x)=V(x)=g(x)$. Since $V\geq g$, we thus only consider the initial state $x\in\mathsf{E}$ satisfying $V(x)-g(x)>0$. Hence, it is enough to show that

$$
a\psi(x)-\mathcal{A}\psi(x)-f(x)\leq 0. \tag{3.2.10}
$$

Case 1. Assume that $x\in\mathsf{E}$ is absorbing. Then by Lemma 3.13, $\psi(x)=V(x)=f(x)/a$. Since $(\mathcal{A},D(\mathcal{A}))$ is an a-subgenerator, applying similar arguments as in the proof of Case 1 for viscosity supersolution, we obtain $\mathcal{A}\psi(x)\geq 0$. Therefore, (3.2.10) is satisfied.

Case 2. Assume that x is not absorbing. Then by Lemma 3.12 and Lemma 3.14, there exists a constant $\Delta>0$ such that for any $\delta\leq\Delta$, we have $\boldsymbol{E}^x[\tau_\delta]<\infty$ and (3.2.6) holds. Since $\psi\in D(\mathcal{A})$ and $\psi\geq V$, we have

$$
\begin{aligned}
V(x)= & \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}f(X(s))\mathrm{d}s+e^{-a\tau_\delta}V(X(\tau_\delta))\Big] \\
\leq & \boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}f(X(s))\mathrm{d}s+e^{-a\tau_\delta}\psi(X(\tau_\delta))\Big] \\
\leq & \psi(x)+\boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}\big(f(X(s))+\mathcal{A}\psi(X(s))-a\psi(X(s))\big)\mathrm{d}s\Big],
\end{aligned} \tag{3.2.11}
$$

where the last inequality follows from the optional sampling theorem since $(\mathcal{A},D(\mathcal{A}))$ is an a-subgenerator. Since $V(x)=\psi(x)$, dividing $\boldsymbol{E}^x\big[\tau_\delta\big]$ on both sides of

(3.2.11), we get

$$0 \leq \frac{\boldsymbol{E}^x\Big[\int_0^{\tau_\delta} e^{-as}\big(f(X(s)) + \mathcal{A}\psi(X(s)) - a\psi(X(s))\big)\mathrm{d}s\Big]}{\boldsymbol{E}^x\big[\tau_\delta\big]}, \tag{3.2.12}$$

for all $\delta \leq \Delta$. Then, since $f, \mathcal{A}\phi$ and ϕ belong to $\mathcal{C}(\mathsf{E})$ and $\boldsymbol{E}^x\big[\tau_\delta\big]$ is bounded, taking $\delta \to 0$, we obtain the desired result. ■

3.3 Uniqueness of viscosity solution for compact state space E

In this section, we prove that the value function is the uniqueness of viscosity solution under the assumption that the state space is compact. In this section, we assume that the state space is compact. Theorem 3.15 gives a comparison principle for viscosity supersolution and subsolution for the compact space. This key result will be needed in the proof of the uniqueness of the viscosity solution associated with $(\mathcal{L}, D(\mathcal{L}))$ (*respectively,* $(\mathcal{G}, D(\mathcal{G}))$) (see Theorem 3.17).

Theorem 3.15. *(Comparison Principle). Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{G})$*. Furthermore, suppose* w_1 *is a viscosity supersolution and* w_2 *is a viscosity subsolution associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to*

$$\min\left(aw - \mathcal{G}w - f, w - g\right) = 0. \tag{3.3.1}$$

Then $w_1 \geq w_2$*.*

Proof. See Section 3.3.1. ■

Remark 3.16. *Since we suppose in the proof that all constant functions belong to* $D(\mathcal{L})$ *or* $D(\mathcal{G})$*, it is natural to assume the compactness of* E*. However, the latter is not a necessary condition to show the uniqueness of the viscosity solution and it will be relaxed in the subsequent sections; see for example Section 3.4 and Proposition 3.24.*

The following theorem constitutes the second main result of this section

Theorem 3.17. *Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{G})$*. Then the value function* (3.1.2) *is the unique viscosity solution associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1)*.*

Proof. The existence follows from Corollary 3.11. Using Theorem 3.15, if there exists another viscosity solution, it must coincide with the value function. ■

3.3.1 Proof of Theorem 3.15

Next we present ingredients for the proof of Theorem 3.15. Let us first recall that the state space E is compact and $D(\mathcal{G})$ (or $D(\mathcal{L})$) contains constant functions. Recall further that $(\mathcal{G}, D(\mathcal{G}))$ is the core of the infinitesimal generator $(\mathcal{L}, D(\mathcal{L}))$.

We prove Theorem 3.15 in three steps. In the first step, we define a notion of classical solution to (3.3.1) and show a partial comparison principle between a classical subsolution (*respectively*, supersolution) and a viscosity supersolution (*respectively*, subsolution). Second, we show that there exists a sequence of classical subsolutions (*respectively*, supersolutions) that converges from below (*respectively*, above) to the value function V defined by (3.1.2). Finally, we use the results from steps 1 and 2 to prove Theorem 3.15.

Step 1

In this step, we first define the notion of classical subsolution (*respectively*, supersolution) to (3.3.1) and then prove a classical comparison theorem.

Definition 3.18. *A function w is a classical subsolution (respectively, supersolution) associated with $(\mathcal{A}, D(\mathcal{A}))$ to (3.3.1), if $w \in D(\mathcal{A})$ and satisfies*

$$\min\left(aw - \mathcal{A}w - f, w - g\right) \le (\ge) 0. \tag{3.3.2}$$

Lemma 3.19. *Suppose Assumption 1 holds. Then a classical subsolution* (*respectively, supersolution*) *associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1) *is a viscosity subsolution* (*respectively, supersolution*) *associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1).

Proof. Let w be a classical subsolution to (3.3.1). By contradiction, assume that w is not a viscosity subsolution to (3.3.1). Then, there exists a function $\phi \in D(\mathcal{G})$ such that $\phi - w$ has a global maximum at x with $(\phi - w)(x) = 0$ and

$$\min\left(a\phi(x) - \mathcal{G}\phi(x) - f(x), w(x) - g(x)\right) > 0. \tag{3.3.3}$$

Since $w - \phi$ has a global nonnegative maximum at x, the positive maximum principle yields $\mathcal{G}(w - \phi)(x) \le 0$, that is, $\mathcal{G}w(x) \le \mathcal{G}\phi(x)$. This together with $w(x) = \phi(x)$ and (3.3.3) gives

$$\min\left(aw(x) - \mathcal{G}w(x) - f(x), w(x) - g(x)\right) > 0,$$

hence contradicting the assumption that w is a classical subsolution to (3.3.1). Therefore w is a viscosity subsolution to (3.3.1). The proof for the supersolution follows the same manner. ∎

We will also need the following partial comparison theorem.

Lemma 3.20. *(Partial Comparison Principle) Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{G})$. *Let* w_1 *be a supersolution associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1) *and* w_2 *be a subsolution associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1), *where one of the solutions is in the classical sense and the other is in the viscosity sense. Then,* $w_1 \geq w_2$.

Proof. Let w_1 be a classical supersolution to (3.3.1) and w_2 be a viscosity subsolution to (3.3.1). Since $D(\mathcal{G}) \subseteq \mathcal{C}_0(\mathsf{E})$, we have that w_1 are in $\mathcal{C}_0(\mathsf{E})$. Since $w_2 \in USC(\mathsf{E})$ and E is compact, there exists $x \in \mathsf{E}$ such that

$$\delta := \sup_{y \in \mathsf{E}} (w_2 - w_1)(y) = (w_2 - w_1)(x).$$

By contradiction, assume that $\delta > 0$ and define w_1^* by

$$w_1^* := w_1 + \delta.$$

Since w_1, δ (as a constant function) are in $D(\mathcal{G})$ and $w_1^* - w_2$ has a global minimum at x with $(w_1^* - w_2)(x) = 0$, it follows that w_1^* is a well defined test function for the viscosity subsolution w_2. Moreover, by the positive maximum principle, we have $\mathcal{G}\delta \leq 0$. Hence,

$$\begin{aligned}\min(aw_1^* - \mathcal{G}w_1^* - f, w_1^* - g)(x) &= \min(a(w_1 + \delta) - \mathcal{G}w_1 - \mathcal{G}\delta - f, w_1 + \delta - g)(x) \\ &\geq \min(aw_1 - \mathcal{G}w_1 - f, w_1 - g)(x) + \min(a, 1)\delta.\end{aligned}$$

Since w_1 is a classical supersolution, we have

$$\min(aw_1 - \mathcal{G}w_1 - f, w_1 - g)(x) \geq 0.$$

Therefore,

$$\min(aw_1^* - \mathcal{G}w_1^* - f, w_1^* - g)(x) \geq \min(a, 1)\delta > 0.$$

This contradicts the fact that w_2 is a viscosity subsolution to (3.3.1). Thus $\sup_{x \in \mathsf{E}}(w_2 - w_1)(x) = \delta \leq 0$, that is, $w_2 \leq w_1$ on E. Similar arguments can be used to show that $w_1 \geq w_2$, if w_1 is a viscosity supersolution to (3.3.1) and w_2 is a classical subsolution to (3.3.1). ■

Corollary 3.21. *(Classical Comparison Principle) Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{G})$. *Let* w_1 *be a classical supersolution and* w_2 *be a classical subsolution associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to (3.3.1). Then,* $w_1 \geq w_2$.

Proof. By Lemma 3.19, we know that a classical supersolution (*respectively*, subsolution) to (3.3.1) is also a viscosity supersolution (*respectively*, subsolution) to (3.3.1). Then, by partial comparison principle, the result follows. ■

Step 2

In this step we first show that there exists a sequence of classical supersolution (*respectively*, subsolution) that converges from above (*respectively*, below) to the value function V.

Lemma 3.22. *Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{L})$*. Then there exists a sequence of classical supersolutions (respectively, subsolutions) associated with* $(\mathcal{L}, D(\mathcal{L}))$ *to* (3.2.2) *that converges to the value function* V *defined by* (3.1.2) *uniformly from the above (respectively, below).*

Proof.

(1) Classical Supersolutions. It follows from Theorem 3.3 that the sequence $\{v_\lambda\}_{\lambda>0} \in D(\mathcal{L})$ defined by (3.1.5) converges uniformly to V from below when $\lambda \to \infty$. Thus, there exists a subsequence $\{\lambda_n\}_{n\in\mathbb{N}^+}$ such that $0 \leq V - v_{\lambda_n} \leq \frac{1}{n}$. Define the sequence $\{w_n\}_{n\in\mathbb{N}^+}$ by

$$w_n := v_{\lambda_n} + \frac{1}{n} \text{ for } n \in \mathbb{N}^+.$$

Then for $n \in \mathbb{N}^+$

$$0 \leq w_n - V = v_{\lambda_n} + \frac{1}{n} - V \leq \frac{1}{n}. \tag{3.3.4}$$

Combining (3.1.5) and (3.3.4) and using the fact that $\mathcal{L}(1/n) \leq 0$ by the positive maximum principle, we obtain

$$\begin{aligned} aw_n - \mathcal{L}w_n - f =& a\left(v_{\lambda_n} + \frac{1}{n}\right) - \mathcal{L}\left(v_{\lambda_n} + \frac{1}{n}\right) - f \\ =& av_{\lambda_n} - \mathcal{L}v_{\lambda_n} - f + \frac{a}{n} - \mathcal{L}\frac{1}{n}, \\ \geq& \lambda_n(g - v_{\lambda_n})^+ + \frac{a}{n} > 0. \end{aligned}$$

Since $w_n - g \geq w_n - V \geq 0$ by (3.3.4), the above inequality implies $\min(aw_n - \mathcal{L}w_n - f, w_n - g) \geq 0$, that is, w_n is a classical supersolution to (3.2.2) by $w_n \in D(\mathcal{L})$. Furthermore, by (3.3.4), $\{w_n\}_{n\in\mathbb{N}^+}$ is a sequence of classical supersolutions to (3.2.2) that converges uniformly to V from above as $n \to \infty$.

(2) Classical Subsolutions. Choose once more the sequence $\{v_\lambda\}_{\lambda>0}$ defined by (3.1.5). For any $\lambda > 0$ or $x \in \mathsf{E}$, one of the following two expressions $v_\lambda(x) - g(x)$ and $\lambda(g(x) - v_\lambda(x))^+$ is non-positive. Then,

$$\min(av_\lambda - \mathcal{L}v_\lambda - f, v_\lambda - g)(x) = \min(\lambda(g - v_\lambda)^+, v_\lambda - g) \leq 0.$$

Hence, $\{v_\lambda\}_{\lambda>0}$ is a sequence of classical subsolutions to (3.2.2) and its uniform convergence from below becomes straightforward by Theorem 3.3. ■

Corollary 3.23. *Suppose Assumption 1 holds. Suppose* E *is compact and the constant function* $1 \in D(\mathcal{G})$*. Then there exists a sequence of classical supersolutions (respectively, subsolutions) associated with* $(\mathcal{G}, D(\mathcal{G}))$ *to* (3.3.1) *that converges to the value function* V *defined by* (3.1.2) *uniformly from above (respectively, below).*

Proof. We know from Lemma 3.22 that there exists a sequence of classical supersolutions associated with $(\mathcal{L}, D(\mathcal{L}))$ to (3.2.2) such that $\{w_n\}_{n\in\mathbb{N}^+}$ satisfies $0 \le w_n - V \le 1/n$ for $n \in \mathbb{N}^+$. Let $\varepsilon > 0$ and choose an integer n_0 such that $n_0 \ge \frac{4}{\varepsilon}$. Set

$$w^{(\varepsilon)} := w_{n_0} + \varepsilon/4.$$

Then,

$$\begin{aligned} w^{(\varepsilon)} - V &= (w^{(\varepsilon)} - w_{n_0}) + (w_{n_0} - V) \le \frac{\varepsilon}{4} + \frac{1}{n_0} \le \frac{\varepsilon}{2}, \qquad (3.3.5)\\ \text{and} \quad w^{(\varepsilon)} - V &= w_{n_0} + \frac{\varepsilon}{4} - V \ge \frac{\varepsilon}{4}. \qquad (3.3.6) \end{aligned}$$

Since w_{n_0} is a classical supersolution to (3.2.2), we have

$$\begin{aligned} \min(aw^{(\varepsilon)} - \mathcal{L}w^{(\varepsilon)} - f, w^{(\varepsilon)} - g) &= \min(a(w_{n_0} + \varepsilon/4) - \mathcal{L}(w_{n_0} + \varepsilon/4) - f, w_{n_0} + \varepsilon/4 - g)\\ &\ge \frac{\min(a,1)\varepsilon}{4} + \min(aw_{n_0} - \mathcal{L}w_{n_0} - f, w_{n_0} - g)\\ &\ge \frac{\min(a,1)\varepsilon}{4}. \qquad (3.3.7) \end{aligned}$$

Therefore, $w^{(\varepsilon)}$ is also a classical supersolution associated with $(\mathcal{L}, D(\mathcal{L}))$ to (3.2.2). Since $(\mathcal{G}, D(\mathcal{G}))$ is the core of $(\mathcal{L}, D(\mathcal{L}))$, it follows that for $w^{(\varepsilon)} \in D(\mathcal{L})$, there exists a sequence $\{u_m^{(\varepsilon)}\}_{m\in\mathbb{N}^+}$ in $D(\mathcal{G})$ such that

$$\|u_m^{(\varepsilon)} - w^{(\varepsilon)}\|_\infty \le \frac{1}{m} \text{ and } \|\mathcal{G}u_m^{(\varepsilon)} - \mathcal{L}w^{(\varepsilon)}\|_\infty \le \frac{1}{m} \text{ for any } m \in \mathbb{N}^+. \qquad (3.3.8)$$

In what follows, we will construct a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of classical supersolution associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1) that converges to V from above. Since $u_m^{(\varepsilon)} \ge$

$w^{(\varepsilon)} - 1/m$ and $\mathcal{G}u_m^{(\varepsilon)} \le \mathcal{L}w^{(\varepsilon)} + 1/m$, we have

$$\begin{aligned}\min(au_m^{(\varepsilon)} - \mathcal{G}u_m^{(\varepsilon)} - f, u_m - g) \ge &\min(a(w^{(\varepsilon)} - \frac{1}{m}) - (\mathcal{L}w^{(\varepsilon)} + \frac{1}{m}) - f, w^{(\varepsilon)} - \frac{1}{m} - g) \\ \ge &- (a+1)\frac{1}{m} + \min(aw^{(\varepsilon)} - \mathcal{L}w - f, w^{(\varepsilon)} - g).\end{aligned} \tag{3.3.9}$$

Choose $m_0 := m_0(\varepsilon) \in \mathbb{N}^+$ such that $m_0 \ge \max(\frac{4}{\varepsilon}, \frac{4(a+1)}{\min(a,1)\varepsilon})$, then for $m \ge m_0$, we have: on the one hand, using (3.3.7) and (3.3.9), $u_m^{(\varepsilon)}$ is a classical supersolution to (3.3.1); on the other hand, using (3.3.5) and (3.3.8)

$$0 \le u_m^{(\varepsilon)} - V = (u_m^{(\varepsilon)} - w) + (w - V) \le \frac{1}{m} + \frac{\varepsilon}{2} \le \varepsilon.$$

Define a new sequence $\{u_\varepsilon\}_{\varepsilon>0}$ by setting $u_\varepsilon := u_{m_0(\varepsilon)}^{(\varepsilon)}$. Then u_ε is a classical supersolution associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1) satisfying $0 \le u_\varepsilon - V \le \varepsilon$ for any arbitrary $\varepsilon > 0$. Therefore, $\{u_\varepsilon\}_{\varepsilon>0}$ converges uniformly to the value function V from above as $\varepsilon \to 0$.

The case of subsolutions can be proved in a similar way. ∎

Step 3

Finally, we prove the comparison principle stated in Theorem 3.15. **Proof of Theorem 3.15.** Define the sets of classical supersolutions and subsolutions associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1) as follows,

$$H_{sup} := \{u \in D(\mathcal{G}); u \text{ is a classical supersolution associated with } (\mathcal{G}, D(\mathcal{G})) \text{ to (3.3.1)}\} \tag{3.3.10}$$

$$H_{sub} := \{v \in D(\mathcal{G}); v \text{ is a classical subsolution associated with } (\mathcal{G}, D(\mathcal{G})) \text{ to (3.3.1)}\}. \tag{3.3.11}$$

Let w_1 be a viscosity supersolution associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1). By Lemma 3.20, it is true that $w_1 \ge u$ for any $u \in H_{sub}$, and then $w_1(x) \ge \sup_{v \in H_{sub}} v(x)$. Similarly, let w_2 be a viscosity subsolution associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1), then, $w_2 \le u$ for any $u \in H_{sup}$ and $w_2(x) \le \inf_{u \in H_{sup}} u(x)$. By Corollary 3.23, there exists a sequence of classical supersolutoins $\{u_n\}_{n\in\mathbb{N}^+}$ (*respectively*, subsolutions $\{v_n\}_{n\in\mathbb{N}^+}$) associated with $(\mathcal{G}, D(\mathcal{G}))$ to (3.3.1) converging uniformly to the value function V from above (*respectively*, below) as $n \to \infty$. Then for any $x \in \mathsf{E}$, we have

$$\begin{aligned} w_1(x) &\geq \sup_{v \in H_{sub}} v(x) \geq \limsup_{n\to\infty} v_n(x) = V(x), \\ w_2(x) &\leq \inf_{u \in H_{sup}} u(x) \leq \liminf_{n\to\infty} u_n(x) = V(x). \end{aligned}$$

Therefore, $w_1 \geq V \geq w_2$. The proof is completed. ■

3.4 Uniqueness of Viscosity Solution for non-compact state space

Both Assumption 1 and compactness condition in Theorem 3.17 give sufficient conditions to prove the existence and uniqueness of the viscosity solution using probabilistic and analytical techniques. However, the compactness of E is not always satisfied for some interesting Feller processes used in practice, for example Lévy processes on $\mathbb{R}^n$ and one dimension diffusions on $[0,\infty)$; see for instance Section 3.6.1.1 and Section 3.6.1.2. Hence, Theorem 3.17 is not immediately applicable for such processes. In addition, since a Feller semigroup is not necessarily conservative, its generator $(\mathcal{L}, D(\mathcal{L}))$ may not have a corresponding Feller process X. In this section, we do not assume the existence of a Feller process (confer conditions (2) and (3) in Assumption 1) and neither do we assume the compactness of E. We first extend the given Feller semigroup on $\mathcal{C}_0(\mathsf{E})$ to a conservative Feller semigroup on $\mathcal{C}(\mathsf{E}_\partial)$. From this we construct an associated Feller process with the aim of characterizing a viscosity solution associated with a core $(\mathcal{G}, D(\mathcal{G}))$ of any infinitesimal generator.

Recall that $\mathsf{E}_\partial := \mathsf{E} \cup \{\partial\}$ is the one point compactification of E. We now extend the Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ on $\mathcal{C}_0(\mathsf{E})$ to a semigroup $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ on $\mathcal{C}(\mathsf{E}_\partial)$ defined by

$$\tilde{\mathcal{P}}_t w(x) := \begin{cases} \mathcal{P}_t(w - w(\partial))|_{\mathsf{E}}(x) + w(\partial) & \text{for any } x \in \mathsf{E}, \\ w(\partial) & \text{otherwise,} \end{cases} \tag{3.4.1}$$

where $w \in \mathcal{C}(\mathsf{E}_\partial)$ and $t \geq 0$. Here $f|_{\mathsf{E}}$ is the restriction of the function f on E. It follows from [Kallenberg, 2006, Lemma 17.13 and 17.14] that $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ is a conservative Feller semigroup. Furthermore by [Blumenthal and Getoor, 2007, Theorem I.9.4], for a conservative Feller semigroup, there always exists a Feller process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ on the state space $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ such that

$$\tilde{\mathcal{P}}_t w(x) := \boldsymbol{E}^x[w(X_t)] \quad \text{for } w \in B(\mathsf{E}) \text{ and } x \in \mathsf{E}_\partial. \tag{3.4.2}$$

This allows us to relate any Feller semigroup on $\mathcal{C}_0(\mathsf{E})$ with a Feller process whose state space E_∂ is the one-point compactification of E. Hence, Theorem 3.17 could also be useful in this case. We first show the relation between the infinitesimal generator of the Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ and that of its extension $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$. To this end, we recall the definition of $\mathcal{C}_*(\mathsf{E})$ as follows:

$$\mathcal{C}_*(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ is converges at infinity}\}.$$

For any $w \in \mathcal{C}_*(\mathsf{E})$, w has a continuous extension $\tilde{w}$ to E_∂.

More precisely, assume that E is not compact, then by one-point compactification technique, E is a dense open subset of E_∂ and w converges to a unique limit C at infinity. Thus, we can define a unique continuous extension $\tilde{w} \in \mathcal{C}(\mathsf{E}_\partial)$ of $w \in \mathcal{C}_*(\mathsf{E})$ by

$$\tilde{w}(x) := \begin{cases} w(x) & \text{for any } x \in \mathsf{E}, \\ C & \text{for } x = \partial. \end{cases} \tag{3.4.3}$$

On the other hand, if E is compact and ∂ is an isolated point, then we simply define its continuous extension of $w \in \mathcal{C}_*(\mathsf{E})$ by

$$\tilde{w}(x) := \begin{cases} w(x) & \text{for } x \in \mathsf{E}, \\ 0 & \text{for } x = \partial. \end{cases} \tag{3.4.4}$$

Therefore, we uniquely define the continuous extension $\tilde{w}(x)$ of $w \in \mathcal{C}_*(\mathsf{E})$ by (3.4.3) or (3.4.4).

3.4.1 Main Results

In this section, we present the main results. Let us now introduce the following operator $(\mathcal{G}^*, D(\mathcal{G}^*))$ defined by

$$\begin{aligned} D(\mathcal{G}^*) &:= \{u \in \mathcal{C}_*(\mathsf{E}); u - \tilde{u}(\partial) \in D(\mathcal{G})\}, \\ \mathcal{G}^* u &:= \mathcal{G}(u - \tilde{u}(\partial)) \text{ for each } u \in D(\mathcal{G}^*). \end{aligned} \tag{3.4.5}$$

Let us mention that when E is compact, it follows from (3.4.4) that $(\mathcal{G}^*, D(\mathcal{G}^*)) = (\mathcal{G}, D(\mathcal{G}))$. The proof of Theorem 3.26 relies on Theorem 3.17 and the following crucial result.

Proposition 3.24. *Suppose that $(\mathcal{G}, D(\mathcal{G}))$ is a core of a Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$, $a > 0$ and $f, g \in \mathcal{C}_*(\mathsf{E})$ (When E is compact, we additionally assume that $1 \in D(\mathcal{G})$.) Then there exists a unique function $w \in \mathcal{C}_*(\mathsf{E})$ with boundary condition $\tilde{w}(\partial) = \max(\tilde{f}(\partial), \tilde{g}(\partial))$ and w is a viscosity solution $w \in \mathcal{C}_*(\mathsf{E})$ associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ defined by (3.4.5) to*

$$\min(aw - \mathcal{G}^* w - f, w - g) = 0. \tag{3.4.6}$$

Moreover, the extension $\tilde{w} \in \mathcal{C}(\mathsf{E}_\partial)$ *is the unique viscosity solution associated with* $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ *defined by* (3.4.13) *to*

$$\min(a\tilde{w} - \tilde{\mathcal{G}}\tilde{w} - \tilde{f}, \tilde{w} - \tilde{g}) = 0, \tag{3.4.7}$$

where $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ *is the core of Feller semigoup* $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ *on* $\mathcal{C}(\mathsf{E}_\partial)$ *defined by (3.4.1).*

Proof. See Section 3.4.2.1. ∎

Remark 3.25. *The above proposition is used to show uniqueness of viscosity solution when the generator is given by a infinitesimal generator of a Feller semigroup rather than a Feller process. Let us emphasize that we need not this Feller semigroup to be conservative nor on a compact state space* E*; see for example Corollary 3.44.*

The main results of this section are the following.

Theorem 3.26. *Suppose Assumption 1 holds. Then the value function* V *defined by* (3.1.2) *is the unique viscosity solution* $w \in \mathcal{C}_b(\mathsf{E})$ *associated with* $(\mathcal{G}^*, D(\mathcal{G}^*))$ *to*

$$\min(aw - \mathcal{G}^* w - f, w - g) = 0, \tag{3.4.8}$$

with $(\mathcal{G}^*, D(\mathcal{G}^*))$ *given by* (3.4.5).

Proof. See Section 3.4.2.2. ∎

Theorem 3.27. *Suppose Assumption 1 holds. Let* $w_1 \in USC(\mathsf{E})$ *and* $w_2 \in LSC(\mathsf{E})$ *are the viscosity subsolution and supersolution associated with* $(\mathcal{G}^*, D(\mathcal{G}^*))$ *to* (4.2.15)*, respectively. If* w_1 *and* w_2 *are bounded from above and below, respectively, then,* $w_1 \leq w_2$.

Remark 3.28. *The operator* $(\mathcal{G}^*, D(\mathcal{G}^*))$ *in Theorem 3.26 always contains the constant function by construction. If one chooses an operator that does not contain this function, then the uniqueness might fail to hold as illustrated in the following example.*

Example 3.29 (Non uniqueness of viscosity solution)**.** *Let* X *be a standard Brownian motion on* $\mathbb{R}$ *and choose* $\left(\frac{1}{2}D_{xx}, \mathcal{C}_c^\infty(\mathbb{R})\right)$ *as its core. By definition, the domain of this operator does not contain constant functions. Set* $f > 0 \in \mathcal{C}_0(\mathbb{R})$ *and* $g = 0$ *in the optimal stopping problem. Then, the value function defined by* (3.1.2) *is reduced to*

$$V(x) := \sup_{\tau\in\mathcal{T}} \boldsymbol{E}^x\Big[\int_0^\tau e^{-as} f(X(s))\,\mathrm{d}s\Big] = J_x(\tau^*) = \mathcal{R}_a f(x) \tag{3.4.9}$$

for $x \in \mathbb{R}$ and the optimal stopping time strategy is $\tau^* = \infty$. By Theorem 3.9, $V = \mathcal{R}_a f \in \mathcal{C}_0(\mathbb{R})$ is a viscosity solution associated with $(\frac{1}{2}D_{xx}, \mathcal{C}_c^\infty(\mathbb{R}))$ to

$$\min(aw - \frac{1}{2}D_{xx}w - f, w) = 0.$$

Let $c > 0$ and set $w = c\mathcal{R}_a f > 0$. We claim that there is no $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\phi - w$ has a global minimum equal 0 at $x_0 \in \mathbb{R}$. Indeed assume that there exists $x_0 \in \mathbb{R}$ such that

$$\phi(x_0) - w(x_0) = 0 \leq \phi(x) - w(x) \text{ for all } x \in \mathbb{R}. \tag{3.4.10}$$

Since ϕ is of compact support, there exists $y_0 \in \mathbb{R}$ such that $\phi(y_0) = 0$. Choose $x = y_0$ then $\phi(y_0) - w(y_0) = -w(y_0) < 0$. This contradict the fact that $\phi - w$ has a global minimum equal 0 at x_0. Since $c > 0$ is chosen arbitrarily, it follows that for every strictly positive function f, the function w defined by $w := c\mathcal{R}_a f > 0$ is a viscosity subsolution.
On the other hand, let $(\mathcal{L}, D(\mathcal{L}))$ be the infinitesimal generator of the standard Brownian motion. Let $c \geq 1$ and set $w = c\mathcal{R}_a f \in D(\mathcal{L})$. Let us show that w is a classical supersolution associated with $(\mathcal{L}, D(\mathcal{L}))$ to

$$\min(aw - \mathcal{L}w - f, w) = 0,$$

Indeed, we have

$$\min(aw - \mathcal{L}w - f, w) = \min(cf - f, \mathcal{R}_a f) \geq 0.$$

The equality follows by (2.2.3) and the inequality follows since $c \geq 1$. Hence by Lemma 3.19, $w = c\mathcal{R}_a f \in D(\mathcal{L})$ is a viscosity supersolution associated with $(\mathcal{L}, D(\mathcal{L}))$. Thus, it is also a viscosity supersolution associated with $(\frac{1}{2}D_{xx}, \mathcal{C}_c^\infty(\mathbb{R}))$. Therefore, for $c \geq 1$ the function $w = c\mathcal{R}_a f$ is a viscosity solution associated with $(\frac{1}{2}D_{xx}, \mathcal{C}_c^\infty(\mathbb{R}))$. Since $c \in [1, \infty)$ is arbitrarily chosen, the uniqueness is not satisfied. □

Remark 3.30. *It is worth mentioning that by Theorem 3.26, the viscosity solution associated with $(\frac{1}{2}D_{xx}, D(\mathcal{G}^*))$ (where $D(\mathcal{G}^*) := \{v \in \mathcal{C}_*(\mathbb{R}); v - \tilde{v}(\partial) \in \mathcal{C}_c^\infty(\mathbb{R})\}$) is unique (see Corollary 3.40).*

3.4.2 Proof of the Main results

3.4.2.1 Proof of Proposition 3.24

Before proving the main results, we need some preliminary results. We start with the following lemma that gives the relation between the infinitesimal generator of the Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ and that of its extension $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$.

Lemma 3.31. *Let $\{\mathcal{P}_t\}_{t\geq 0}$ be a Feller semigroup on $\mathcal{C}_0(\mathsf{E})$, whose infinitesimal generator is $(\mathcal{L}, D(\mathcal{L}))$ with a core $(\mathcal{G}, D(\mathcal{G}))$. Given the Feller semigroup $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ defined by* (3.4.1)*, its infinitesimal generator $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ satisfies*

$$\tilde{\mathcal{L}}w = \widetilde{\mathcal{L}((w - w(\partial))|_{\mathsf{E}})} \text{ for each } w \in D(\tilde{\mathcal{L}}), \tag{3.4.11}$$

with

$$D(\tilde{\mathcal{L}}) = \{w \in \mathcal{C}(\mathsf{E}_\partial); (w - w(\partial))|_{\mathsf{E}} \in D(\mathcal{L})\}. \tag{3.4.12}$$

Furthermore, suppose $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ is the restriction of $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ on $D(\tilde{\mathcal{G}})$ with

$$\begin{aligned} D(\tilde{\mathcal{G}}) &:= \{w \in \mathcal{C}(\mathsf{E}_\partial); (w - w(\partial))|_{\mathsf{E}} \in D(\mathcal{G})\} \\ \tilde{\mathcal{G}}w &:= \widetilde{\mathcal{G}((w - w(\partial))|_{\mathsf{E}})} \text{ for } w \in D(\tilde{\mathcal{G}}) \end{aligned} \tag{3.4.13}$$

Then $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ is also the core of the Feller semigroup $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$.

Proof. Let $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ be the Feller semigroup defined by (3.4.1), that is for any $w \in \mathcal{C}(\mathsf{E}_\partial)$,

$$\tilde{\mathcal{P}}_t w := w(\partial) + \widetilde{\mathcal{P}_t((w - w(\partial))|_{\mathsf{E}})}. \tag{3.4.14}$$

By Definition 2.4, its infinitesimal generator $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ can be defined by:

$$\tilde{\mathcal{L}}w := \lim_{t\to 0^+} \frac{\tilde{\mathcal{P}}_t w - w}{t} \text{ for each } w \in D(\tilde{\mathcal{L}}), \tag{3.4.15}$$

$$D(\tilde{\mathcal{L}}) := \{w \in \mathcal{C}(\mathsf{E}_\partial); \lim_{t\to 0^+} \frac{\tilde{\mathcal{P}}_t w - w}{t} \text{ exists in } \mathcal{C}(\mathsf{E}_\partial)\}. \tag{3.4.16}$$

Let D_0 be a domain defined by

$$D_0 := \{w \in \mathcal{C}(\mathsf{E}_\partial); (w - w(\partial))|_{\mathsf{E}} \in D(\mathcal{L})\}. \tag{3.4.17}$$

We show that $D(\tilde{\mathcal{L}}) = D_0$ by double inclusion. We first prove that $D_0 \subseteq D(\tilde{\mathcal{L}})$. Let $w \in D_0 \subseteq \mathcal{C}(\mathsf{E}_\partial)$. We have by (3.4.14) restricted on E that

$$\lim_{t\to 0^+} \frac{(\tilde{\mathcal{P}}_t w)|_{\mathsf{E}} - w|_{\mathsf{E}}}{t} = \lim_{t\to 0^+} \frac{\mathcal{P}_t((w - w(\partial))|_E) + w(\partial) - w|_{\mathsf{E}}}{t}. \tag{3.4.18}$$

Since $w \in D_0$, we have $(w - w(\partial))|_{\mathsf{E}} \in D(\mathcal{L})$, then the limit on the right hand side of (3.4.18) exists in $\mathcal{C}_0(\mathsf{E})$ and we can write

$$\lim_{t\to 0^+}\frac{(\tilde{\mathcal{P}}_t w)|_{\mathsf{E}} - w|_{\mathsf{E}}}{t} = \mathcal{L}((w - w(\partial))|_E) \in \mathcal{C}_0(\mathsf{E}). \tag{3.4.19}$$

In addition, using (3.4.14) and the fact that $(w-w(\partial))|_{\mathsf{E}} \in \mathcal{C}_0(\mathsf{E})$, $\tilde{\mathcal{P}}_t w(\partial) = w(\partial)$ for all $t \geq 0$. Hence, we know that

$$\lim_{t\to 0^+}\frac{\tilde{\mathcal{P}}_t w(\partial) - w(\partial)}{t} = 0. \tag{3.4.20}$$

Putting (3.4.19) and (3.4.20) together yields for any $w \in D_0$, $\lim_{t\to 0^+}\frac{\tilde{\mathcal{P}}_t w - w}{t}$ exists in $\mathcal{C}(\mathsf{E}_\partial)$ and by the definition of the extension, we get

$$\lim_{t\to 0^+}\frac{\tilde{\mathcal{P}}_t w - w}{t} = \widetilde{\mathcal{L}((w - w(\partial))|_{\mathsf{E}})} \text{ exists in } \mathcal{C}(\mathsf{E}_\partial) \text{ for any } w \in D_0. \tag{3.4.21}$$

Thus, $D_0 \subseteq D(\tilde{\mathcal{L}})$.

Let us now prove that $D(\tilde{\mathcal{L}}) \subseteq D_0$. Choose $w \in D(\tilde{\mathcal{L}})$. Then, since for such w, the limit of (3.4.15) exists in $\mathcal{C}(\mathsf{E}_\partial)$, it follows that the limit on the right hand side of (3.4.18) also exists. In addition, using (3.4.16) and (3.4.20) respectively, we have $\tilde{\mathcal{L}}w \in \mathcal{C}(\mathsf{E}_\partial)$ and $\tilde{\mathcal{L}}w(\partial) = 0$ and thus the limit

$$\lim_{t\to 0^+}\frac{\mathcal{P}_t((w - w(\partial))|_E) + (w - w(\partial))|_E}{t} \text{ exists in } \mathcal{C}_0(\mathsf{E}).$$

Therefore, due to the fact that $(w-w(\partial))|_{\mathsf{E}} \in \mathcal{C}_0(\mathsf{E})$, we have $(w-w(\partial))|_{\mathsf{E}} \in D(\mathcal{L})$ which means $w \in D_0$. We can conclude that $D(\tilde{\mathcal{L}}) = D_0$ and $\tilde{\mathcal{L}}$ is given by (3.4.21), that is (3.4.11) and (3.4.12) are proved.

Then, it is reasonable to define the restriction of $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ on $D(\tilde{\mathcal{G}})$ by (3.4.13).

Let us now show that $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ is the core of $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$. Suppose that there is a sequence $\{w_n\}_{n\in\mathbb{N}^+}$ in $D(\tilde{\mathcal{G}})$ satisfying $w_n \to u$ and $\tilde{\mathcal{G}}w_n \to v$ uniformly in $\mathcal{C}(\mathsf{E}_\partial)$. It is enough to prove that $u \in D(\tilde{\mathcal{L}})$ and $v = \tilde{\mathcal{L}}u$. Using (3.4.13), the sequence $\{w_n^*\}_{n\in\mathbb{N}^+}$ defined by

$$w_n^* := (w_n - w_n(\partial))|_{\mathsf{E}} \text{ for } n \in \mathbb{N}^+$$

belongs to $D(\mathcal{G})$ and satisfies $w_n^* \to (u - u(\partial))|_{\mathsf{E}}$ and $\mathcal{G}w_n^* \to v|_{\mathsf{E}}$ uniformly in $\mathcal{C}_0(\mathsf{E})$. In addition, since $(\mathcal{G}, D(\mathcal{G}))$ is the core of $(\mathcal{L}, D(\mathcal{L}))$, it follows that $(u - u(\partial))|_{\mathsf{E}} \in D(\mathcal{L})$ and $v|_{\mathsf{E}} = \mathcal{L}((u - u(\partial))|_{\mathsf{E}})$. Therefore, using (3.4.15), $u \in D(\tilde{\mathcal{L}})$ and $v = \tilde{\mathcal{L}}u$. The proof is completed.

■

Since $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ defined by (3.4.1) is a conservative Feller semigroup, we know from [Blumenthal and Getoor, 2007, Theorem I.9.4] that there exists a corresponding Feller process $\tilde{X}$ whose transition semigroup is $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ with the compact state space E_∂. $\tilde{X}$ is also a standard Markov process. Let us now define the value function $\tilde{V}$ of $\tilde{X}$ by

$$\tilde{V}(x) := \sup_\tau \tilde{\boldsymbol{E}}^x\Big[\int_0^\tau e^{-as}\tilde{f}(\tilde{X}_s)\,\mathrm{d}s + e^{-a\tau}\tilde{g}(\tilde{X}(\tau))\Big] \text{ for } x \in \mathsf{E}_\partial. \tag{3.4.22}$$

One can check that all the conditions in Assumption 1 are fulfilled. In fact, E_∂ is compact; using Lemma 3.31, $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ defined by (3.4.13) is the core of the Feller process $\tilde{X}$ and $f, g \in \mathcal{C}_*(\mathsf{E})$ implies $\tilde{f}, \tilde{g} \in \mathcal{C}(\mathsf{E}_\partial)$. Then, by Theorem 3.17, the value function $\tilde{V} \in \mathcal{C}(\mathsf{E}_\partial)$ defined by (3.4.22) is the unique viscosity solution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to

$$\min(a\tilde{w} - \tilde{\mathcal{G}}\tilde{w} - \tilde{f}, \tilde{w} - \tilde{g}) = 0. \tag{3.4.23}$$

Lemma 3.32. *Suppose the assumptions in Proposition 3.24 hold. Assume that $w \in USC(\mathsf{E})$ (respectively, $LSC(\mathsf{E})$) is a viscosity subsolution (respectively, supersolution) associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (4.2.15). Define the extension $\bar{w}$ on E_∂ by*

$$\bar{w} := \begin{cases} w(x) & \text{for } x \in \mathsf{E} \\ \max(\frac{\tilde{f}(\partial)}{a}, \tilde{g}(\partial)) & \text{for } x = \partial. \end{cases} \tag{3.4.24}$$

If $\bar{w} \in USC(\mathsf{E}_\partial)$ (respectively, $LSC(\mathsf{E}_\partial)$), then $\bar{w}$ is a viscosity subsolution (respectively, supersolution) associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23).

Proof. Let $w \in USC(\mathsf{E})$ be a viscosity subsolution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6). We want to show that $\bar{w} \in USC(\mathsf{E}_\partial)$ is also a viscosity subsolution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23). Let $\phi \in D(\tilde{\mathcal{G}})$ such that $\phi - \bar{w}$ has a global minimum at x in E_∂ with $\phi(x) = \bar{w}(x)$, we want to show that

$$\min\left(a\phi(x) - \tilde{\mathcal{G}}\phi(x) - \tilde{f}(x), \phi(x) - \tilde{g}(x)\right) \leq 0. \tag{3.4.25}$$

We distinguish two cases:
(a) Assume that $x = \partial$ (an absorbing point). Then, $\tilde{\mathcal{G}}\phi(\partial) = 0$ for all $\phi \in D(\tilde{\mathcal{G}})$. In addition, since $\bar{w}(\partial) \leq \max(\tilde{f}(\partial)/a, \tilde{g}(\partial))$, (3.4.25) is satisfied.
(b) Assume that $x \in \mathsf{E}$. Define $\phi^* \in C_*(\mathsf{E})$ by $\phi^* := \phi|_{\mathsf{E}}$. Since $\phi \in D(\tilde{\mathcal{G}})$, it follows from (3.4.13) that $\phi^* - \phi(\partial) \in D(\mathcal{G})$. In addition, we claim that $\phi^* \in D(\mathcal{G}^*)$.

To see this, we first assume that E is not compact. Then $\phi(\partial) = \tilde{\phi}^*(\partial)$ and thus $\phi^* - \tilde{\phi}^*(\partial) = (\phi - \phi(\partial))|_{\mathsf{E}} \in D(\mathcal{G})$ (since $\phi \in D(\tilde{\mathcal{G}})$). Hence $\phi^* \in D(\mathcal{G}^*)$. Next, we assume that E is compact. In this case, $\phi \in D(\tilde{\mathcal{G}})$ means $\phi \in C(\mathsf{E}_\partial)$ and $(\phi - \phi(\partial))|_{\mathsf{E}} \in D(\mathcal{G})$, that is, $\phi|_{\mathsf{E}} - \phi(\partial) \in D(\mathcal{G})$. Using the fact that $1 \in D(\mathcal{G})$, we obtain $\phi^* := \phi|_{\mathsf{E}} \in D(\mathcal{G})$. Therefore, since $\tilde{\phi}^*(\partial) = 0$ by the compactness of E and $\phi^* \in D(\mathcal{G})$, it follows from (3.4.5) that $\phi^* \in D(\mathcal{G}^*)$. The claim is thus proved.

Next, recall that $\phi - \bar{w}$ has a global minimum at x in E_∂ with $\phi(x) = \bar{w}(x)$. Hence, using $\phi^* := \phi|_{\mathsf{E}}$ and $w = \bar{w}|_{\mathsf{E}}$, it follows that $\phi^* - w$ has a global minimum at x in E with $\phi^*(x) = w(x)$. Combining this with the fact that $\phi^* \in D(\mathcal{G}^*)$, and since w is viscosity subsolution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6), we have

$$\min\left(a\phi^*(x) - \mathcal{G}^*\phi^*(x) - f(x), \phi^*(x) - g(x)\right) \le 0. \tag{3.4.26}$$

Since $f = \tilde{f}|_{\mathsf{E}}, g = \tilde{g}|_{\mathsf{E}}$ and $\phi^* = \phi|_{\mathsf{E}}$, in order to prove that (3.4.25) holds when $x \in \mathsf{E}$, it is enough to show that

$$\tilde{\mathcal{G}}\phi(x) \ge \mathcal{G}^*\phi^*(x). \tag{3.4.27}$$

It follows from (3.4.13) (*respectively*, (3.4.5)) that $\tilde{\mathcal{G}}\tilde{\phi}^*(x) = \mathcal{G}(\tilde{\phi}^* - \tilde{\phi}^*(\partial))|_{\mathsf{E}}(x)$ (respectively, $\mathcal{G}^*\phi^*(x) = \mathcal{G}(\phi^* - \tilde{\phi}^*(\partial))(x))$ and thus $\tilde{\mathcal{G}}\tilde{\phi}^*(x) = \mathcal{G}^*\phi^*(x)$. That is, (3.4.27) becomes

$$\tilde{\mathcal{G}}\phi(x) \ge \tilde{\mathcal{G}}\tilde{\phi}^*(x). \tag{3.4.28}$$

We again distinguish two cases.

(i) Assume that E is not compact. By the uniqueness of the extension, we have $\phi = \tilde{\phi}^*$ and $\tilde{\mathcal{G}}\phi(x) = \tilde{\mathcal{G}}\tilde{\phi}^*$.

(ii) Assume that E is compact. By the definition of $\tilde{\phi}^*$ given by (3.4.4), we have $\tilde{\phi}^*(y) = \phi(y)$ for any $y \in \mathsf{E}$ and $\tilde{\phi}^*(\partial) = 0$. In addition, since $\phi - \bar{w}$ has a global minimum at x in E_∂ and $w \in \mathcal{C}_0(\mathsf{E})$, we have $\phi(y) \ge \bar{w}(y)$ for any $y \in \mathsf{E}_\partial$ and thus $\phi(\partial) \ge \bar{w}(\partial) = \max(\tilde{f}(\partial)/a, \tilde{g}(\partial)) = 0 = \tilde{\phi}^*(\partial)$, since E is compact. This indicates that $\tilde{\phi}^* - \phi$ has a positive maximum equal 0 at x in E_∂. By Lemma 3.31, $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ is the core of $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$. Then from Theorem 2.7, $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ satisfies the positive maximum principle and thus $\tilde{\mathcal{G}}\phi(x) \ge \tilde{\mathcal{G}}\tilde{\phi}^*(x)$.

The viscosity supersolution can be proved in a similar way. ■

Proof of Proposition 3.24. **(1)** We will prove that $V := \tilde{V}|_{\mathsf{E}}$ is a viscosity solution in $\mathcal{C}_*(\mathsf{E})$ associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6). We first prove that V is a viscosity subsolution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6). Let $x \in \mathsf{E}$ and $\psi \in D(\mathcal{G}^*)$ such that $\psi - v_e$ has a global minimum at x with $\psi(x) = V(x)$. There are two cases:

(i) Suppose that E is compact. Then $\tilde{\psi}(\partial) - V(\partial) = 0$.

(ii) Suppose that E in not compact. Since E is a dense open subset of E_∂, we

have $\tilde{\psi}(\partial) - \tilde{V}(\partial) \geq 0$.
It follows that $\tilde{\psi} - \tilde{V}$ has a global minimum at x in E_∂ with $\tilde{\psi}(x) = \tilde{V}(x)$. Moreover, since $\psi \in D(\mathcal{G}^*)$, then by the definition (3.4.5) of $D(\mathcal{G}^*)$, we have $\psi - \tilde{\psi}(\partial) \in D(\mathcal{G})$. Hence, $(\tilde{\psi} - \tilde{\psi}(\partial))|_{\mathsf{E}} \in D(\mathcal{G})$. Therefore, $\tilde{\psi} \in D(\tilde{\mathcal{G}})$ by the definition (3.4.13) of $D(\tilde{\mathcal{G}})$. Since the value function $\tilde{V}$ defined by (3.4.22) is a viscosity subsolution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23), we have

$$\min\left(a\tilde{\psi}(x) - \tilde{\mathcal{G}}\tilde{\psi}(x) - \tilde{f}(x), \tilde{\psi}(x) - \tilde{g}(x)\right) \leq 0.$$

Furthermore, since $x \in \mathsf{E}$, using (3.4.13), we have $\tilde{\mathcal{G}}\tilde{\psi}(x) = \mathcal{G}(\tilde{\psi} - \tilde{\psi}(\partial))|_{\mathsf{E}}(x)$, and using (3.4.5), we have $\mathcal{G}^*\psi(x) = \mathcal{G}(\psi - \tilde{\psi}(\partial))(x)$. Hence, $\tilde{\mathcal{G}}\tilde{\psi}(x) = \mathcal{G}^*\psi(x)$. Therefore,

$$\min\left(a\psi(x) - \mathcal{G}^*\psi(x) - f(x), \psi(x) - g(x)\right) \leq 0.$$

We can also prove in a similar way that V is a viscosity supersolution. The existence is then proved, that is, $V = \tilde{V}|_{\mathsf{E}}$ is a viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6).

(2) Next, we show that $V|_{\mathsf{E}}$ is the unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$. The idea here is to prove that if $w \in \mathcal{C}_*(\mathsf{E})$ is a viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6), then $\tilde{w} \in \mathcal{C}(\mathsf{E}_\partial)$ is a viscosity solution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23). Hence, the result will follow since the viscosity solution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23) is unique. Using Lemma 3.32, if $w \in \mathcal{C}_*(\mathsf{E})$ is a viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6), its extension $\bar{w}$ is the unique viscosity solution associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$ to (3.4.23) which is the value function V defined by (3.4.22). This completes the proof of the uniqueness and the proposition. ∎

3.4.2.2 Proof of Theorem 3.26

For E compact, since $\mathcal{C}_b(\mathsf{E}) = \mathcal{C}_0(\mathsf{E})$, the existence and uniqueness follow Theorem 3.17. Thus, we only need to consider the case E not compact.

Existence: Using Theorem 3.9, the viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (4.2.15) is the value function provided that $(\mathcal{G}^*, D(\mathcal{G}^*))$ is an a-generator.

Let us show that $(\mathcal{G}^*, D(\mathcal{G}^*))$ is an a-generator. By Dynkin's formula and the argument preceding Corollary 3.11, we have $(\mathcal{G}, D(\mathcal{G}))$ is an a-generator. Let us consider the restriction of $(\mathcal{G}^*, D(\mathcal{G}^*))$ to the space of constant functions. Since E is not compact, using (3.4.5), we have $\mathcal{G}^*1 = \mathcal{G}0 = 0$ by (3.4.5). Hence it follows that $\{S_1(t)\}_{t\geq 0}$ given by (3.1.4) (with $w = 1$) is a $(\mathcal{F}_t, P^x)$ uniformly integrable martingale for $a > 0$ and thus $(\mathcal{G}^*, D(\mathcal{G}^*))$ is an a-generator.

Uniqueness: For the uniqueness, let $\{\phi_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{C}_0(\mathsf{E})$ converging pointwisely to the constant function 1. By Dini's theorem, $\{\phi_n\}_{n\in N}$ converges to 1 locally uniformly. Let $C \geq \max(\|f\|_\infty, \|g\|_\infty)$. Define

$f_n^- := \phi_n \cdot (f + C) - C$ and $g_n^- := \phi_n \cdot (g + C) - C$. Then, since $\{\phi_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}_0(\mathsf{E})$ is increasing, $\{f_n^-\}_{n\in\mathbb{N}}$ and $\{g_n^-\}_{n\in\mathbb{N}}$ are in $\mathcal{C}_*(\mathsf{E})$ and increasing.

Let $w \in \mathcal{C}_b(\mathsf{E})$ be a viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (4.2.15), which satisfies $w \geq -C$, and define

$$w_n^-(x) := \sup_\tau \boldsymbol{E}^x\big[\int_0^\tau e^{-as} f_n^-(X(s))\mathrm{d}s + e^{-a\tau} g_n^-(X(\tau))\big].$$

By the existence proof, w_n^- is a viscosity solution to

$$\min(aw_n^- - \mathcal{G}^* w_n^- - f_n^-, w_n^- - g_n^-) = 0. \tag{3.4.29}$$

Since $f \geq f_n^-$ and $g \geq g_n^-$ in E, w is a viscosity supersolution to (3.4.29). Therefore, by Lemma 3.32, $w \geq w_n^-$ for all $n \in \mathbb{N}$.

Similarly, let $f_n^+ := \phi_n \cdot (f - C) + C$ and $g_n^+ := \phi_n \cdot (g - C) + C$ and w_n^+ is

$$w_n^+(x) := \sup_\tau \boldsymbol{E}^x\big[\int_0^\tau e^{-as} f_n^+(X(s))\mathrm{d}s + e^{-a\tau} g_n^+(X(\tau))\big].$$

It is the viscosity solution to

$$\min(aw_n^+ - \mathcal{G}^* w_n^+ - f_n^+, w_n^+ - g_n^+) = 0. \tag{3.4.30}$$

Then, similarly, since $w \leq C$, by Lemma 3.32, w is the viscosity subsolution to and then $w \leq w_n^+$.

Therefore, since $w_n^+ \geq w \geq w_n^-$ for all $n \in \mathbb{N}$, to prove the uniqueness, it is enough to show that $\lim_{n\to} w_n^+(x) = \lim_{n\to} w_n^-(x) = V(x)$. We have the following inequalities,

$$\begin{aligned}
V(x) - w_n^-(x) \leq &\sup_\tau \boldsymbol{E}^x\Bigg[\int_0^\tau e^{-as}(1 - \phi_n(X(s)))(f(X(s)) - \|f\|_\infty)\mathrm{d}s \\
&\qquad + e^{-a\tau}(1 - \phi_n(X(\tau)))](g(X(\tau)) - \|g\|_\infty)\Bigg] \\
\leq &\sup_\tau \boldsymbol{E}^x\Bigg[\int_0^\tau e^{-as}(1 - \phi_n(X(s)))\|f\|_\infty \mathrm{d}s + e^{-a\tau}(1 - \phi_n(X(\tau)))\|g\|_\infty\Bigg] \\
\leq &C\big[\mathcal{R}_a(1 - \phi_n)(x) + \sup_\tau \boldsymbol{E}^x\big[e^{-a\tau}(1 - \phi_n(X(\tau)))\big]\big].
\end{aligned}$$

By [Schilling, 1998, Theorem 3.2], we know that $\mathcal{R}_a(1-\phi_n)$ converges to 0 locally uniformly. Then, we only need to prove u_n converges to 0 locally uniformly, with $u_n(x) := \sup_\tau \boldsymbol{E}^x\big[e^{-a\tau}(1 - \phi_n(X(\tau)))$.

This can be proved by the following property shown in [Palczewski and Stettner, 2010, Proposition 2.1]. For any compact set $K \subseteq \mathsf{E}$, $T > 0$ and $\varepsilon > 0$, there exists a compact set $L_\varepsilon \subseteq \mathsf{E}$ such that

$$\sup_{x\in K} \boldsymbol{P}^x(X(s) \notin L_\varepsilon \text{ for some } t \in [0,T]) < \varepsilon. \tag{3.4.31}$$

Therefore, for any $\mathcal{F}_t$-stopping time τ, we have for all $x \in K$

$$\begin{aligned}\boldsymbol{E}^x\left[e^{-a\tau}(1-\phi_n(X(\tau)))\right] =& \boldsymbol{E}^x\left[e^{-a\tau}(1-\phi_n(X(\tau)))\mathbf{1}_{\tau<T} + e^{-a\tau}(1-\phi_n(X(\tau)))\mathbf{1}_{\tau>T}\right]\\ \leq& \boldsymbol{E}^x\left[e^{-a\tau}(1-\phi_n(X(\tau)))\mathbf{1}_{\tau<T}\right] + e^{-aT}\\ \leq& \boldsymbol{E}^x\left[e^{-a\tau}(1-\phi_n(X(\tau)))\mathbf{1}_{\tau<T}\mathbf{1}_{\{X(s)\notin L_\varepsilon \text{ for some } t\in[0,T]\}}\right]\\ &+ \boldsymbol{E}^x\left[e^{-a\tau}(1-\phi_n(X(\tau)))\mathbf{1}_{\tau<T}\mathbf{1}_{\{X(s)\in L_\varepsilon \text{ for all } t\in[0,T]\}}\right] + e^{-aT}\\ \leq& \varepsilon + \sup_{x\in L_\varepsilon}(1-\phi_n(x)) + e^{-aT}.\end{aligned}$$

Since L_ε is compact and $\{\phi_n\}_{n\in\mathbb{N}}$ converges to 1 locally uniformly, $\sup_{x\in L_\varepsilon}(1-\phi_n(x))$ converges to 0 as $n\to\infty$. Since ε, K and T are all arbitrarily chosen, u_n converges to 0 locally uniformly. Therefore, we know that $\{w_n^-\}_{n\in\mathbb{N}}$ converges to V locally uniformly. Similarly, we have $\{w_n^+\}_{n\in\mathbb{N}}$ converges to V locally uniformly. Then, this completes the proof of the uniqueness. □

Corollary 3.33. *Suppose Assumption 1 holds and define $(\mathcal{G}^*, D(\mathcal{G}^*))$ by (3.4.5). Then there exists a sequence of classical supersolutions (respectivley subsolution) $\{v_n\}_{n\in\mathbb{N}^+}$ associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ that converges to the viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6) from the above (respectively, below).*

Proof. Choose $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$, $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$, X and V as Proposition 3.24. Since E_∂ is compact, by Corollary 3.23, there exists a sequence of classical supersolutions (respectivley subsolution) $\{\tilde{v}_n\}_{n\in\mathbb{N}^+}$ associated with $(\tilde{\mathcal{G}}, D(\tilde{\mathcal{G}}))$, that is, $\tilde{v}_n \in D(\tilde{\mathcal{G}})$ and

$$\min(a\tilde{v}_n - \tilde{\mathcal{G}}\tilde{v}_n - \tilde{f}, \tilde{v}_n - \tilde{g}) \geq (\leq) 0 \text{ for } n \in \mathbb{N}^+, \tag{3.4.32}$$

and $\{\tilde{v}_n\}_{n\in\mathbb{N}^+}$ converges uniformly in $\mathcal{C}(\mathsf{E}_\partial)$ to V from the above (*respectively*, below). Let us now introduce the sequence $\{v_n\}_{n\in\mathbb{N}^+}$ define by

$$v_n := \tilde{v}_n|_{\mathsf{E}} \text{ for } n \in \mathbb{N}^+.$$

v_n converges uniformly to $V|_{\mathsf{E}}$ in $\mathcal{C}_0(\mathsf{E})$ from above (*respectively*, below). Furthermore, we know from Proposition 3.24 that $V|_{\mathsf{E}}$ is the unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (3.4.6), and thus the proof is completed if we show that for $n \in \mathbb{N}^+$, $v_n \in D(\mathcal{G}^*)$ and

$$\min(av_n - \mathcal{G}^*v_n - f, v_n - g) \geq (\leq) 0 \text{ for } n \in \mathbb{N}^+. \tag{3.4.33}$$

Using (3.4.5) and (3.4.13), $\tilde{\mathcal{G}}\tilde{v}_n(x) = \mathcal{G}(\tilde{v}_n - \tilde{v}_n(\partial)|_{\mathsf{E}})(x) = \mathcal{G}^* v_n(x)$ for all $x \in \mathsf{E}$ and $n \in \mathbb{N}^+$. Combining this with (3.4.32), we have (3.4.33). ■

3.5 Structure of the optimal stopping value functions

In this section, we related the viscosity solution to some existing results, using martingale approach. First, we introduce some preliminary lemmas.

Lemma 3.34. *Given $u \in LSC(\mathsf{E})$ (respectively, $u \in USC(\mathsf{E})$), define the process $\{M(t)\}_{t\geq 0}$ by*

$$M(t) := e^{-at}u(X(t)) + \int_0^t e^{-as}f(X(s))\mathrm{d}s. \tag{3.5.1}$$

Suppose there exists an open subset $\mathcal{O} \subseteq \mathsf{E}$ such that $\{M(t \wedge \tau_{\mathcal{O}})\}_{t\geq 0}$ is a $(\mathcal{F}_t, \boldsymbol{P}^x)$ uniformly integrable supermartingale (respectively, submartingale) for all $x \in \mathcal{O}$. Then, the following claims hold.

1. *For all $\phi \in D(\mathcal{G}^*)$ such that $\phi - u$ has a global maximum (respectively, minimum) at $x_0 \in \mathcal{O}$ with $\phi(x_0) = u(x_0)$,*

$$a\phi(x_0) - \mathcal{G}^*\phi(x_0) - f(x_0) \leq (\geq) 0. \tag{3.5.2}$$

2. *Additionally, suppose there exists a subset $K_0 \subseteq \mathsf{E}$ such that X satisfies $\boldsymbol{P}^{x_0}\big[X(\tau_{\mathcal{O}}) \in K_0\big] = 1$ for some $x_0 \in \mathcal{O}$. Then for all $\psi \in D(\mathcal{G}^*)$ such that $\psi - w$ has a maximum (respectively, minimum) in K_0 at x_0 with $\psi(x_0) = w(x_0)$, we have*

$$a\phi(x_0) - \mathcal{G}^*\psi(x_0) - f(x_0) \leq (\geq) 0. \tag{3.5.3}$$

Proof. The proof is similar as the proof of Theorem 3.9. Here, we only prove the statement (2) since the statement (1) follows when $K_0 = \mathsf{E}$ in the statement (2). Let $\psi \in D(\mathcal{G}^*)$ such that $\psi - u$ has a maximum (*respectively*, minimum) in K_0 at x_0 with $\psi(x_0) = u(x_0)$. Let the process $\{S(t)\}_{t\geq 0}$ be defined by

$$S(t) := e^{-at}\psi(X(t)) + \int_0^t e^{-as}(a\psi(X(s)) - \mathcal{G}^*\psi(X(s)))\mathrm{d}s. \tag{3.5.4}$$

By Dynkin formula, since $\psi \in D(\mathcal{G}^*)$, $\{S(t)\}_{t\geq 0}$ is a $(\mathcal{F}_t, \boldsymbol{P}^{x_0})$ uniformly integrable martingale. We first assume x_0 is a point not absorbing. Let $\delta > 0$ and

τ_δ defined by (3.2.3). Since $\{M(t\wedge\tau_{\mathcal{O}})\}_{t\geq 0}$ is a $(\mathcal{F}_t, \boldsymbol{P}^{x_0})$ uniformly integrable supermartingale, we have

$$\begin{aligned}
u(x_0) &\geq \boldsymbol{E}^{x_0}\left[\int_0^{\tau_\delta} e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau_\delta} u(X(\tau_\delta))\right] \\
&\geq \boldsymbol{E}^{x_0}\left[\int_0^{\tau_\delta} e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau_\delta} \psi(X(\tau_\delta))\right] \\
&\geq \boldsymbol{E}^{x_0}\left[\int_0^{\tau_\delta} e^{-as} (f(X(s)) + \mathcal{G}^*\psi(X(s)) - a\psi(X(s)))\mathrm{d}s\right] + \psi(x),
\end{aligned}$$

where the last inequality follows from the optional stopping theorem. Then, similar as (3.2.8) in Theorem 3.9, (3.5.3) is proved. For the case the non-absorbing point x_0, we can prove it similarly as Theorem 3.9. ■

Corollary 3.35. *Let $u \in LSC(\mathsf{E})$ be bounded from the below. Suppose that its corresponding process $\{M(t)\}_{t\geq 0}$ defined by* (3.5.1) *is a supermartingale. If $u \geq g$, then u is a viscosity supersolution to*

$$\min(aw - \mathcal{G}^*w - f, w - g) = 0 \tag{3.5.5}$$

and $u \geq V$, where V is the value function defined by (3.1.2).

Proof. Since $\{M(t)\}_{t\geq 0}$ is a supermaringale, by Lemma 3.34, u is a viscosity supersolution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to

$$aw - \mathcal{G}^*w - f = 0.$$

Since $u \geq g$, u is also a viscosity supersolution to

$$\min(aw - \mathcal{G}^*w - f, w - g) = 0.$$

By the comparison principle (see Theorem 3.27), we have $u \geq V$. ■

Corollary 3.36. *Let $u \in USC(\mathsf{E})$ be bounded from above. Suppose there exists an open subset $\mathcal{O} \in \mathsf{E}$ such that its corresponding process $\{M(t\wedge\tau_{\mathcal{O}})\}_{t\geq 0}$ defined by* (3.5.1) *is a submartingale.*

1. *If $u(x) \leq g(x)$ for all $x \notin \mathcal{O}$, then $u(y) \leq V(y)$ for all $y \in \mathcal{O}$, where V is the value function defined by* (3.1.2).
2. *Additionally, suppose there exists a subset $\bar{\mathcal{O}} \subseteq K_0 \subseteq \mathsf{E}$ such that X satisfies $\boldsymbol{P}^x\big[X(\tau_{\mathcal{O}}) \in K_0\big] = 1$ for all $x \in \mathcal{O}$. If $u(x) \leq g(x)$ for all $x \in K_0 \setminus \mathcal{O}$, then $u(y) \leq V(y)$ for all $y \in \mathcal{O}$.*

Proof. As in Lemma 3.34, we give the proof of (2) and (1) follows setting $K_0 = \mathsf{E}$. First define a function u_- as

$$u_-(x) := \begin{cases} u(x) & \text{for } x \in \mathcal{O} \\ u(x) \wedge g(x) & \text{for } x \notin \mathcal{O}. \end{cases}$$

Since $\mathcal{O}$ is an open subset with $\bar{\mathcal{O}} \subseteq K_0$, g is a continuous function and $u(x) \leq g(x)$ for $x \in K_0 \setminus \mathcal{O}$, we have $u_- \in USC(\mathsf{E})$.

Similarly with Corollary 3.35, by Lemma 3.34, u_- is a viscosity subsolution to

$$aw(x) - \mathcal{G}^* w(x) - f(x) = 0 \text{ for } x \in \mathcal{O}.$$

Since $u_-(x) \leq g(x)$ for all $x \notin \mathcal{O}$, then u is a viscosity subsolution to

$$\min(aw - \mathcal{G}^* w - f, w - g) = 0.$$

By the comparison principle (see Theorem 3.27), we have $u_- \leq V$ and then $u(x) \leq V(x)$ for all $x \in \mathcal{O}$.

■

Combing Corollary 3.35 and Corollary 3.36, the following result suggests that the value function characterized in the proof of [Alili and Kyprianou, 2005, Theorem 3.1] coincides with the viscosity solution to (3.4.6).

Theorem 3.37. *Suppose that there exists $u \in \mathcal{C}_b(\mathsf{E})$ and an open subset $\mathcal{O} \subseteq \mathsf{E}$ satisfying the following statements. Additionally suppose that there exists a subset $\bar{\mathcal{O}} \subseteq K_0 \subseteq \mathsf{E}$ such that X satisfies that $\boldsymbol{P}^x\big[X(\tau_{\mathcal{O}}) \in K_0\big] = 1$ for all $x \in \mathcal{O}$.*

1. *$\{M(t \wedge \tau_{\mathcal{O}})\}_{t\geq 0}$ is a uniformly integrable martingale,*
2. *$\{M(t)\}_{t\geq 0}$ is a supermartingale,*
3. *$u \geq g$ and $u(x) = g(x)$ for $x \in K_0 \setminus \mathcal{O}$,*

Then, $u(x) = V(x)$ for all $x \in \mathcal{O}$.

The above theorem gives a classical method to find the optimal stopping value function using the martingale characterization. It is traditionally used for one-dimensional process to find explicit solution for optimal stopping for diffusion. We should mention that the martingale approach usually does not require the continuity and boundedness of the reward functions f and g. (See for example Beibel and Lerche [2001].)

3.6 Applications

3.6.1 Viscosity properties of value functions for optimal stopping problems

In this section, we apply our results to study viscosity properties for optimal stopping problems for some processes satisfying Assumption 1 and whose core fulfills the conditions of our main theorems. Let us mention that many traditional processes studied in the literature satisfy those assumptions. Thus, we revisit the optimal stopping using viscosity approach developed in the thesis. To our knowledge, optimal stopping problems for Brownian motion jumping at boundary and semi-Markov process have not been studied in the literature using viscosity approach. Recall that the objective function

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^\tau e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau} g(X(\tau)) \right]. \tag{3.6.1}$$

as given by (3.1.2). Let E be a space to be determined in each example. In this section, we always assume that

$$a > 0 \text{ and } f, g \in \mathcal{C}_b(\mathsf{E}).$$

We will first use Theorem 3.9 to show that the value function given by (3.1.2) is a viscosity solution. Let us start with Lévy processes on the state space $\mathsf{E} = \mathbb{R}^n$.

3.6.1.1 Lévy Processes

Here, we assume that $X = \{X(t)\}_{t\geq 0}$ is a Lévy process on $\mathsf{E} = \mathbb{R}^n$. It is known (see for example [Kallenberg, 2006, Theorem 17.10]) that $X = \{X(t)\}_{t\geq 0}$ is a Feller process. Its core $(\mathcal{G}_{Levy}, D(\mathcal{G}_{Levy}))$ is given by

$$\begin{aligned}\mathcal{G}_{Levy} w(x) =& l \cdot \nabla w(x) + \frac{1}{2} div\, Q\, \nabla w(x) \\ &+ \int_{\mathbb{R}^n \setminus \{0\}} \big(w(x+y) - w(x) - \nabla w(x) \cdot y \mathbf{1}_{|y|<1} \big)\, \nu(\mathrm{d}y),\end{aligned} \tag{3.6.2}$$

for $x \in \mathbb{R}^n$ and $w \in D(\mathcal{G}_{Levy}) := \mathcal{C}_0^\infty(\mathbb{R}^n)$, where $l \in \mathbb{R}^n$ is a vector, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix, ν is a positive Radon measure satisfying $\int_{\mathbb{R}^n\setminus\{0\}} \min(|y|^2, 1)\nu(\mathrm{d}y) < \infty$ and $\mathcal{C}_0^\infty(\mathbb{R}^n)$ denotes the space of all infinitely differentiable functions and itself and all its derivatives belong to $\mathcal{C}_0(\mathbb{R}^n)$. We have the following result from Theorem 3.26.

Proposition 3.38. *Assume that $X = \{X(t)\}_{t\geq 0}$ is a Lévy process whose core $(\mathcal{G}_{Levy}, D(\mathcal{G}_{Levy}))$ is described above. Then the value function V given* (3.6.1) *is the unique viscosity solution $w \in \mathcal{C}_b(\mathbb{R}^n)$ associated with $(\mathcal{G}^*_{Levy}, D(\mathcal{G}^*_{Levy}))$ to*

$$\min(aw - \mathcal{G}^*_{Levy}w - f, w - g) = 0, \tag{3.6.3}$$

where $D(\mathcal{G}^*_{Levy}) = \{v \in \mathcal{C}_*(\mathbb{R}^n); v - \tilde{v}(\partial) \in \mathcal{C}^\infty_0(\mathbb{R}^n)\}$.

Remark 3.39. *Similar optimal stopping problem was studied in Alili and Kyprianou [2005]; Mordecki [2002]. In particular, the authors look at perpetual put options for one dimensional Lévy process with $f = 0$ and $g(x) = (K - e^{\beta x})^+$, where $K > 0$ and $\beta > 0$. More precisely, the value function has the following form*

$$V(x) := \sup_\tau \boldsymbol{E}^x\big[e^{-a\tau}(K - e^{\beta X(\tau)})^+\big]. \tag{3.6.4}$$

Let us note that Alili and Kyprianou [2005] used a martingale approach similar to Theorem 3.37 to show the value function is solution to a martingale problem. Alternatively, we can use Proposition 3.38 to show that the value function is the unique viscosity solution to the associated HJB equation.

Let us now assume that the process $X = \{B(t)\}_{t\geq 0}$ is a one dimensional *standard Brownian motion*, that is, a Feller process with state space $\mathsf{E} = \mathbb{R}$ and core $(\mathcal{G}_{BM}, D(\mathcal{G}_{BM}))$ given by

$$\begin{aligned} D(\mathcal{G}_{BM}) &:= \{u \in \mathcal{C}_0(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R});\ D_x u, D_{xx}u \in \mathcal{C}_0(\mathbb{R})\}, \\ \mathcal{G}_{BM}u(x) &:= \frac{1}{2}D_{xx}u(x) \text{ for } x \in \mathbb{R}. \end{aligned} \tag{3.6.5}$$

Theorem 3.9 gives us the freedom to choose larger domains than $D_0(\mathcal{G}^{BM})$, for example,

$$D(\mathcal{G}^*_{BM}) :=\{u \in \mathcal{C}_*(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}); D_x u, D_{xx}u \in \mathcal{C}_0(\mathbb{R})\}, \tag{3.6.6}$$

$$D(\mathcal{G}^{(b)}_{BM}) :=\{u \in \mathcal{C}_b(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}); D_x u, D_{xx}u \in \mathcal{C}_b(\mathbb{R})\}, \tag{3.6.7}$$

$$\begin{aligned} D(\mathcal{G}^{(p)}_{BM}) :=&\{u \in \mathcal{C}^2(\mathbb{R}); D_{xx}u \in \mathcal{C}_b(\mathbb{R}) \text{ and there exists } K > 0 \\ &\text{such that } |u(x)|^2 \leq K(1 + |x|^2) \text{ for all } x \in \mathbb{R}\}. \end{aligned} \tag{3.6.8}$$

Using Theorem 3.9 and Theorem 3.26, we have the following result:

Corollary 3.40. *Assume that $X = \{B(t)\}_{t\geq 0}$ is a one dimensional standard Brownian motion. Then the value function V given by* (3.6.1) *is the unique viscosity solution $w \in \mathcal{C}_0(\mathbb{R})$ associated with $(\mathcal{G}_{BM}, D(\mathcal{G}^*_{BM}))$ (respectively, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(b)}_{BM}))$,*

$(\mathcal{G}_{BM}, D(\mathcal{G}^{(p)}_{BM})))$ *to*

$$\min(aw - \mathcal{G}_{BM}w - f, w - g) = 0. \tag{3.6.9}$$

Proof. Let us first observe that $(\mathcal{G}_{BM}, D(\mathcal{G}^*_{BM}))$ corresponds to $(\mathcal{G}^*, D(\mathcal{G}^*))$ and $(\mathcal{G}_{BM}, D(\mathcal{G}_{BM}))$ corresponds to $(\mathcal{G}, D(\mathcal{G}))$ in Theorem 3.26. Let $w \in (\mathcal{G}^{BM}, D_*(\mathcal{G}^{BM}))$ (*respectively*, $(\mathcal{G}^{BM}, D_b(\mathcal{G}^{BM})), (\mathcal{G}^{BM}, D_p(\mathcal{G}^{BM})))$. Using Itô's formula, the process $\{S_w(t)\}_{t\geq 0}$ given by

$$S_w(t) := w(X_0) - e^{-at}w(X(t)) - \int_0^t e^{-as}\Big(aw(X(s)) - \frac{1}{2}D_{xx}w(X(s))\Big)\mathrm{d}s \text{ for } t \geq 0$$

is a $(\mathcal{F}_t, \boldsymbol{P}^x)$-uniformly integrable martingale for $a > 0$ and $x \in \mathbb{R}$. Using Definition 3.2 the operator $(\mathcal{G}_{BM}, D(\mathcal{G}^*_{BM}))$ (*respectively*, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(b)}_{BM})), (\mathcal{G}^{BM}, D(\mathcal{G}^{(p)}_{BM})))$ is an a-generator. Hence by Theorem 3.9, the value function V defined by (3.6.1) is a viscosity solution associated with $(\mathcal{G}^{BM}, D(\mathcal{G}^*_{BM}))$ (*respectively*, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(b)}_{BM}))$, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(p)}_{BM})))$. The uniqueness follows from Theorem 3.26 since $(\mathcal{G}_{BM}, D(\mathcal{G}^*_{BM}))$ (*respectively*, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(b)}_{BM}))$, $(\mathcal{G}_{BM}, D(\mathcal{G}^{(p)}_{BM})))$ corresponds to $(\mathcal{A}, D(\mathcal{A}))$. ■

In the next section we consider examples of one dimensional diffusion processes on the positive half line $\mathsf{E} = [0, \infty)$ that behave like a standard Brownian motion with different boundary behaviours at boundary 0.

3.6.1.2 Diffusion on $\mathsf{E} = [0, \infty)$

Let $X = \{X(t)\}_{t\geq 0}$ be a diffusion process on $[0, \infty)$. Then the generator $(\mathcal{G}_{BC}, D(\mathcal{G}_{BC}))$ of $X = \{X(t)\}_{t\geq 0}$ is given by

$$\begin{aligned} D(\mathcal{G}_{BC}) &:= \{u \in \mathcal{C}_0([0, \infty)) \cap \mathcal{C}^2([0, \infty));\ D_x u, D_{xx}u \in \mathcal{C}_0([0, \infty))\}, \\ \mathcal{G}_{BC}u(x) &:= \frac{1}{2}D_{xx}u(x) \text{ for } x \in [0, \infty). \end{aligned} \tag{3.6.10}$$

The operator $(\mathcal{G}_{BC}, D(\mathcal{G}_{BC}))$ does not satisfy the positive maximum principle at 0 unless we add some appropriate conditions at boundary 0. Let us consider the following processes with appropriate domain

1. *Reflected Brownian motion*: $D(\mathcal{G}_{ref}) := \{u \in D(\mathcal{G}_{BC});\ D_x u(0) = 0\}$;
2. *Sticking Brownian motion*: $D(\mathcal{G}_{stk}) := \{u \in D(\mathcal{G}_{BC});\ D_{xx}u(0) = 0\}$;
3. *Sticky reflecting Brownian motion*: $D(\mathcal{G}_{stkref}) := \{u \in D(\mathcal{G}_{BC});\ D_{xx}u(0) = cD_x u(0)\}$, where $c \in (0, \infty)$.
4. *Brownian motion with jump at the boundary*: $D(\mathcal{G}_{jump}) := \{u \in D(\mathcal{G}_{BC});\ D_{xx}u(0) = \lambda \int_{[0,\infty)}(u(0) - u(y))\mu(\mathrm{d}y)\}$, where $\lambda > 0$ and μ is a probability measure.

We have the following result from Theorem 3.9 and Theorem 3.26:

Proposition 3.41. *Assume that $X = \{X(t)\}_{t\geq 0}$ is a reflected Brownian motion (respectively, sticking Brownian motion, sticky reflecting Brownian motion). Then the value function V given by* (3.6.1) *is a unique viscosity solution in $\mathcal{C}_b(\mathsf{E})$ associated with $(\mathcal{G}^{BC}, D(\mathcal{G}^*_{ref}))$ (respectively, $(\mathcal{G}^{BC}, D(\mathcal{G}^*_{stk}))$, $(\mathcal{G}^{BC}, D(\mathcal{G}^*_{stkref}))$), $(\mathcal{G}^{BC}, D(\mathcal{G}^*_{jump}))$).*

Proof. It follows from the fact that the above processes are Feller processes. ■

Now, consider the reflected Brownian motion and define

$$D(\mathcal{G}^+_{ref}) := \{u \in \mathcal{C}_b([0,\infty)) \cap \mathcal{C}^2([0,\infty)); D_x u, D_{xx}u \in \mathcal{C}_b([0,\infty)) \text{ and } D_x u(0) \geq 0\},$$
$$D(\mathcal{G}^-_{ref}) := \{u \in \mathcal{C}_b([0,\infty)) \cap \mathcal{C}^2([0,\infty)); D_x u, D_{xx}u \in \mathcal{C}_b([0,\infty)) \text{ and } D_x u(0) \leq 0\}.$$

Corollary 3.42. *Assume that $X = \{X(t)\}_{t\geq 0}$ is a reflected Brownian motion. Then the value function V given by* (3.1.2) *is the unique function in $\mathcal{C}_0(\mathbb{R}^+)$ which is both a viscosity supersolution associated with $(\mathcal{G}^{BC}, D^+_{ref}(\mathcal{G}^{BC}))$ and a viscosity subsolution associated with $(\mathcal{G}^{BC}, D^-_{ref}(\mathcal{G}^{BC}))$.*

Proof. Since $w \in D(\mathcal{G}^+_{ref})$ (*respectively*, $D(\mathcal{G}^-_{ref})$), the process $\{S_w(t)\}_{t\geq 0}$ given by

$$S_w(t) = w(X_0) - e^{-at}w(X(t)) - \int_0^t e^{-as}\left(aw(X(s)) - \mathcal{G}^{BC}w(X(s))\right) \mathrm{d}s \text{ for } t \geq 0$$

is a $(\mathcal{F}_t, \boldsymbol{P}^x)$ uniformly integrable supermartingale (*respectively*, submartingale). Hence, Theorem 3.9 suggests that the value function defined by (3.1.2) is a viscosity supersolution (*respectively*, subsolution) associated with $(\mathcal{G}_{BC}, D(\mathcal{G}^+_{ref}))$ (*respectively*, $(\mathcal{G}_{BC}, D(\mathcal{G}^-_{ref}))$). As for the uniqueness, we only need to show that it holds for the operator $(\mathcal{G}_{BC}, D(\mathcal{G}^*_{ref}))$, where

$$D(\mathcal{G}^*_{ref}) = \{u \in \mathcal{C}_*([0,\infty)) \cap \mathcal{C}^2([0,\infty));\ D_x u, D_{xx}u \in \mathcal{C}_0(\mathbb{R}^+) \text{ and } Du(0) = 0\}.$$

This follows from Theorem 3.26. Therefore, it leads to the desired result, since $(\mathcal{G}_{BC}, D(\mathcal{G}^*_{ref}))$ can be seen as the restriction of $(\mathcal{G}_{BC}, D(\mathcal{G}^+_{ref}) \cap D(\mathcal{G}^-_{ref}))$ on $D(\mathcal{G}^*_{ref})$. ■

Remark 3.43. *In the above example, we consider the simplest cases of standard Brownian motion with the state space $[0,\infty)$. More generally, Feller Feller [1952, 1954, 1957] constructs Markov processes up to a specific regular boundary point 0 with the boundary condition given by*

$$c_1 w(0) - c_2 Dw(0) + c_3 D_{xx}w(0) = 0,$$

for $c_1, c_2, c_3 \geq 0$ *with* $c_1 + c_2 + c_3 = 1$. *However, we have not considered the cases in which* $c_1 > 0$, *for example, the "Dirichlet condition"* $w(0) = 0$, *or the "Robin condition"* $c_1 w(0) - c_2 Dw(0) = 0$. *The reason is that when* $c_1 > 0$ *the above Markov processes may be killed upon reaching* 0. *This does not coincide with Definition 2.3 of Feller process. Nevertheless, our method is still applicable as demonstrated in the following example.*

Let $(\mathcal{G}_{kill}, D(\mathcal{G}_{kill}))$ be an operator defined by

$$\begin{aligned} D(\mathcal{G}_{kill}) &:= \{u \in \mathcal{C}_0((0,\infty)) \cap \mathcal{C}^2((0,\infty));\ D_x u, D_{xx} u \in \mathcal{C}_0((0,\infty))\}, \\ \mathcal{G}_{kill} u(x) &:= \frac{1}{2} D_{xx} u(x) \text{ for } x \in (0,\infty). \end{aligned} \tag{3.6.11}$$

Using Proposition 3.24, we have the corollary below:

Corollary 3.44. *Suppose* $f, g \in \mathcal{C}_0((0,\infty))$ *and* $a > 0$. *Then there exists a unique viscosity solution* $w \in \mathcal{C}_0((0,\infty))$ *associated with* $(\mathcal{G}_{kill}, D(\mathcal{G}^*_{kill}))$ *to*

$$\min(aw - \mathcal{G}^*_{kill} w - f, w - g) = 0,$$

where $D(\mathcal{G}^*_{kill}) = \{u \in \mathcal{C}_*((0,\infty)) \cap \mathcal{C}^2((0,\infty));\ D_{xx} u \in \mathcal{C}_0((0,\infty))\}$.

Proof. It is known (see for example Feller [1954]) that $(\mathcal{G}_{kill}, D(\mathcal{G}_{kill}))$ is the core of a Feller semigroup. Hence, the result follows from Proposition 3.24. ■

Remark 3.45. *Assume that* $X = \{X(t)\}_{t \geq 0}$ *is a standard Brownian motion. We show in Chapter 5 that under additional assumptions, the unique viscosity solution given in Corollary 3.44 is the value function to the following optimal stopping problem:*

$$V(x) = \sup_\tau \boldsymbol{E}^x \left[\int_0^{\tau \wedge \tau_0} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \mathbf{1}_{\tau < \tau_0} \right] \textit{ for } x \in (0,\infty),$$

where $\tau_0 := \inf\{t > 0; X(t) \notin (0,\infty)\}$.

In the next section, we wish to establish viscosity properties of the value function of the optimal stopping problem (3.6.1), when X is a diffusion with piecewise coefficients. Such problem with discontinuous function f and $g = 0$ was studied in Belomestny et al. [2010]; Rüschendorf and Urusov [2008] using a "modified" free boundary approach. Note in addition that the definition of viscosity solution given in [Belomestny et al., 2010, Definiton 4.2 and 4.3] does not ensure that the value function is the unique solution. In this chapter, assuming that $f, g \in \mathcal{C}_b(\mathsf{E})$ and using different definition of viscosity solution, we show the viscosity property of the value function.

3.6.1.3 Diffusion with piecewise coefficients

We start by constructing a diffusion process $X = \{X(t)\}_{t\geq 0}$ with piecewise coefficients. Let σ, ρ and μ be three bounded real valued measurable functions. Suppose that $\sigma|_{\mathbb{R}\setminus J} \in \mathcal{C}_b^1(\mathbb{R}\setminus J)$ and $\mu|_{\mathbb{R}\setminus J}, \rho|_{\mathbb{R}\setminus J} \in \mathcal{C}_b(\mathbb{R}\setminus J)$, where J is a set in $\mathbb{R}$ without cluster points and contains all the discontinuous points of the functions σ, μ and ρ. In addition, suppose that there exists $\lambda > 0$ such that $\sigma, \mu > \lambda$. We know from [Lejay et al., 2015, Propositions 2.1, 2.2 and 2.6] that there exists a Feller process X with continuous paths whose infinitesimal generator is given by

$$\begin{aligned} D(\mathcal{G}_{pw}) := \{&w \in \mathcal{C}_0(\mathbb{R});\ D_x u, D_{xx}u \text{ exists in } \mathbb{R}\setminus J, \mathcal{G}_{pw}u \in \mathcal{C}_0(\mathbb{R}) \\ &\text{and } \sigma(x^-)D_x u(x^-) = \sigma(x^+)D_x u(x^+) \text{ for all } x \in J\}. \end{aligned}$$

$$\mathcal{G}_{pw}u(x) := \begin{cases} \frac{\rho(x)}{2} D_x(\sigma(x)D_x w)x(x) + \mu x() D_x u(x_x) & \text{for } x \in \mathbb{R}\setminus J, \\ \frac{\rho(x)}{2} D_x(\sigma D_x u)_x((x)x^+) + \mu D_x u(x^+) & \text{for } x \in J, \end{cases}$$

As a consequence of Theorem 3.26, we have the following result.

Corollary 3.46. *Let $X = \{X(t)\}_{t\geq 0}$ be Feller process whose core $(\mathcal{G}^{pw}, D(\mathcal{G}^{pw}))$ is given by in* (3.6.12)*. Then the value function V given by* (3.6.1) *is the unique viscosity solution $w \in \mathcal{C}_b(\mathbb{R})$ associated with $(\mathcal{G}^{pw}, D_*(\mathcal{G}^{pw}))$, where*

$$\begin{aligned} D(\mathcal{G}_{pw}^*) :=&\{u \in \mathcal{C}_*(\mathbb{R});\ D_x u, D_{xx}u \textit{ exists in } \mathbb{R}\setminus J, \mathcal{G}_{pw}u \in \mathcal{C}_0(\mathbb{R}) \\ &\textit{and } \sigma(x^-)D_x u(x^-) = \sigma(x^+)D_x u(x^+) \textit{ for all } x \in J\}. \end{aligned} \tag{3.6.12}$$

In particular, [Revuz and Yor, 2013, Chapter VII, Exercise 1.23] provides an example of *Skew Brownian motion* with parameter $\beta \in (0,1)$. Heuristically speaking, it is constructed by a Brownian motion reflected at zero which enters the positive half line with probability $\frac{\beta+1}{2}$ (*respectively*, the negative half with probability $\frac{1-\beta}{2}$) when it reaches zero. Its core is given by

$$\begin{aligned} D(\mathcal{G}_{skew}) := \{&u \in \mathcal{C}_0(\mathbb{R});\ D_x u, D_{xx}u \text{ exists in } \mathbb{R}\setminus\{0\} \text{ and converges to 0 at infinity}, \\ &D_{xx}w(0^-) = D_{xx}w(0^+) \text{ and } \beta D_x w(0^+) = (1-\beta)D_x w(0^-)\}, \end{aligned}$$

$$\mathcal{G}_{skew}u(x) := \begin{cases} \frac{1}{2}D_{xx}u(x) & \text{for } x \in \mathbb{R}\setminus\{0\}, \\ \frac{1}{2}D_{xx}u(0^+) & \text{for } x = 0. \end{cases} \tag{3.6.13}$$

Again, Theorem 3.26 yields the following result

Corollary 3.47. *Let $X = \{X(t)\}_{t\geq 0}$ be a skew Brownian motion with parameter $\beta \in (0,1)$. Then the value function* (3.1.2) *of the stopping problem* (3.1.1)-(3.1.2) *is the unique viscosity solution $w \in \mathcal{C}_b(\mathbb{R})$ associated with $(\mathcal{G}^*_{skew}, D(\mathcal{G}^*_{skew}))$, where*

$$\begin{aligned} D(\mathcal{G}^*_{skew}) :=&\{u \in \mathcal{C}_*(\mathbb{R});\ D_x u, D_{xx} u \text{ exists in } \mathbb{R}\setminus\{0\} \text{ and converges to } 0 \text{ at infinity},\\ &D_{xx}u(0^-) = D_{xx}u(0^+) \text{ and } \beta D_x u(0^+) = (1-\beta)D_x u(0^-)\}. \end{aligned} \tag{3.6.14}$$

Remark 3.48. *Observe that $D(\mathcal{G}^*_{skew})$ in the above example does not contain any smooth function unless its derivative is equal to 0. This means that if one wants to show that a function has the viscosity property at 0, test functions ϕ as described in Definition 3.1 are continuous but are not smooth at 0. This leads to additional technical difficulty in the proof of the uniqueness when using the traditional method. This is due to the fact that this method is based on smoothness of test function and properties of elliptic or parabolic differential equations.*

3.6.2 Perturbation

Perturbation is a powerful method to transform a known Feller process to a new Feller process. We first introduce the following lemma which enables to construct the Feller semigroup using perturbation.

Lemma 3.49. *Böttcher et al. [2013] Let $(\mathcal{G}, D(\mathcal{G}))$ be the infinitesimal generator of some Feller semigroup on $\mathcal{C}_0(\mathsf{E})$. Assume that $B : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ and B is bounded, that is, there exists $C > 0$ such that $\sup_{u\in\mathcal{C}_0(\mathsf{E})} \frac{\|Bu\|_\infty}{\|u\|_\infty} \le C$. Additionally suppose $(B, \mathcal{C}_0(\mathsf{E}))$ satisfies the positive maximum principle. Then, $(\mathcal{L}+B, D(\mathcal{L}))$ is the infinitesimal generator of some Feller semigroup on $\mathcal{C}_0(\mathsf{E})$.*

Using this method, we provide constructions of Feller processes via perturbation. The first example is the Feller process with large jumps.

3.6.2.1 Compound Poisson Operator

) Let $\{X(t)\}_{t\ge 0}$ be a Feller process with the state space state space $[0,\infty)$ and core given by $(\mathcal{G}, D(\mathcal{G}))$. Define a bounded operator B by

$$Bu(x) := \lambda \int_0^\infty (u(x) - u(x-y))\mathrm{d}\mu(y), \tag{3.6.15}$$

where μ is a probability distribution function defined on $(0,\infty)$ and λ is the intensity parameter. Then by Lemma 3.49, $(\mathcal{G} + B, D(\mathcal{G}^*))$ is some Feller process $\{Y(t)\}_{t\ge 0}$. For example, let $\{B(t)\}_{t\ge 0}$ be a standard Brownian motion and $\mu(x) = 1 - e^{-\gamma x}$ be the distribution function of an exponential random variable with parameter γ. Let $\{X_b(t)\}_{t\ge 0}$ be a compound Poisson process with the intensity

$\lambda > 0$ and the jump height following an exponential distribution with parameter γ. Then, in this case, one can choose $\{Y(t), \mathcal{F}_t^Y\}_{t\geq 0}$ as

$$Y(t) = Y(0) + B(t) + X_b(t) \text{ for } t \geq 0 \tag{3.6.16}$$

with core $(\mathcal{G}_{ref}+B, D(\mathcal{G}_{ref}))$, where $\mathcal{F}_t^Y$ is the natural filtration of $\{Y(t)\}_{t\geq 0}$. Thus $\{Y(t)\}$ is still a Feller process. Hence viscosity solution approach can be used to characterise the value function of the optimal stopping problem of $\{Y(t)\}_{t\geq 0}$. Next, we wish to study properties of the value function for an optimal stopping problem for a semi-Markov process.

3.6.2.2 Semi-Markov Process

Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identical (i.i.d.) random variables with cumulative density distribution function P. Additionally, let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d random variables defined on $\mathbb{R}$ with distribution function F. Let $S_n := \sum_{i=1}^n T_i$ for $n = 0, 1, \ldots$ and the renewal process $N(t) := \max\{n; S_n \leq t\}$. Let $\{X(t)\}_{t\geq 0}$ be

$$X(t) := x + \sum_{i=1}^{N(t)} Y_i \text{ for } t \geq 0, \tag{3.6.17}$$

where x is the initial state. For example when the interarrival time is the exponential distribution, $\{X(t)\}_{t\geq 0}$ is a compound Poisson distribution which is a Markov process. However, if the interarrival time does not follow the exponential distribution, $\{X_t\}_{t\geq 0}$ is not a Markov process but a semi-Markov process. We want to analyze the optimal stopping problem of

$$V_{semi}(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^\tau e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \Big], \tag{3.6.18}$$

where $a > 0$ and $f, g \in \mathcal{C}_b(\mathbb{R})$.

Remark 3.50. *Optimal stopping problems of semi-Markov process has be studied in Boshuizen and Gouweleeuw [1993]; Muciek [2002]. The work Boshuizen and Gouweleeuw [1993] provides several application of semi-Markov processes in real life, for example, job search and shock model (for more detail, see [Boshuizen and Gouweleeuw, 1993, Section 1]. In this section, we want to solve optimal stopping problems using viscosity approach instead of the iterative one as in Boshuizen and Gouweleeuw [1993]; Muciek [2002].*

Assume that P is an absolutely continuous function and p is its continuous density function on $[0, \infty)$. Define $Q(x) := p(x)/(1 - P(x))$ for $x \in [0, \infty)$. In addition, assume that $\lim_{x\to\infty} \frac{p(x)}{1-P(x)} = C$. Then, Q has a continuous extension $\bar{Q}$ on $[0,\infty]$. Examples are:

1. *Mixture exponential distribution*: $P(x) := \sum_{i=1}^m w_i(1-e^{-\lambda_i x})$, where $\sum_{i=1}^m w_i = 1$, $w_i > 0$, $\lambda_i > 0$ and m is some positive integral. The density function is $p(x) = \sum_{i=1}^m w_i \lambda_i e^{-\lambda_i x}$. Therefore, $Q(x) = \frac{p(x)}{1-P(x)}$ and $\lim_{x\to\infty} Q(x) = \min_{i=1,2,\ldots,m} \lambda_i$.

2. Generalized beta prime distribution: let $P(x) := \frac{x}{1+x}$ and $p(x) := \frac{1}{(1+x)^2}$. Then, $Q(x) = \frac{1}{1+x}$ for $[0,\infty)$.

Let $\{\xi(t)\}_{t\geq 0}$ be the time from the last jumps (for example if S_n is the time of the last jump at time t, $\xi(t) = t - S_n$). Then, the two dimesional process $\{\xi(t), X(t)\}_{t\geq 0}$ is a Markov process (see for example [Skorohod et al., 1979, Lemma 2, p290]). Its infinitesimal generator is defined as follows.

$$D(\mathcal{G}) := \{u \in \mathcal{C}_0([0,\infty]\times\mathbb{R}); u(\cdot,x) \in \mathcal{C}^1([0,\infty]) \text{ and } Du(\infty) = 0 \text{ for all } x \in \mathbb{R}\},$$
$$\mathcal{G}u(s,x) := D_s u(s,x) + Q(s)\int_{\mathbb{R}} (u(0,x+y) - u(s,x))\mathrm{d}F(y) \text{ for } s \in [0,\infty] \text{ and } x \in \mathbb{R}, \tag{3.6.19}$$

where $Q(s)$ is the hazard rate of the distribution P of inter-arrival time T_i.

Proposition 3.51. *Assume that X is a semi-Markov process defined by* (3.6.17).

1. *There exists a unique viscosity solution $w \in \mathcal{C}_b([0,\infty]\times\mathbb{R})$ associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ defined by* (3.6.19) *to*
$$\min(aw - \mathcal{G}^* w - \bar{f}, w - \bar{g}) = 0, \tag{3.6.20}$$
where $\bar{f}(s,x) = f(x)$ and $\bar{g}(s,x) = g(x)$ for all $s \in [0,\infty]$ and $x \in \mathbb{R}$.

2. *The value function can be characterized by $V(x) = w(0,x)$.*

Proof. First, we prove that (3.6.19) is an infinitesimal generator of some Feller semigroup. Since $(D_s, D(\mathcal{G}))$ is a generator of some Feller semigroup, by Lemma 3.49, we only need to prove: (i) B is defines from $\mathcal{C}_0(\mathsf{E})$ to $\mathcal{C}_0(\mathsf{E})$, (ii) B is bounded and (iii) B satisfies the positive maximum principle, where

$$Bu(s,x) := Q(s)\int_{\mathbb{R}} (u(0,x+y) - u(s,x))\mathrm{d}F(y) \text{ for } s \in [0,\infty] \text{ and } x \in \mathbb{R}. \tag{3.6.21}$$

Let $u \in \mathcal{C}_0([0,\infty]\times\mathbb{R})$. We have

(i) Since $\int_R u_0(x+y)\mathrm{d}y \in \mathcal{C}_0(\mathbb{R})$ where $u_0(x) := u(0,x)$ for $x \in \mathbb{R}$, $\int_{\mathbb{R}}(u(0,x+y) - u(s,x))\mathrm{d}F(y) = \int_{\mathbb{R}} u(0,x+y)\mathrm{d}y - u(s,x)$ implies that $B : \mathcal{C}_0([0,\infty] \times \mathbb{R}) \to \mathcal{C}_0([0,\infty] \times \mathbb{R})$ from the fact that $Q \in \mathcal{C}_b([0,\infty])$ and $Q \geq 0$.

(ii) Since $|\int_{\mathbb{R}} u(0,x+y)\mathrm{d}y - u(s,x)| \leq 2\|u\|_\infty$ and Q is bounded, we know B is bounded.

(iii) If (s_0,x_0) is the global maximum point and $u(s_0,x_0) \geq 0$, $Bu(s_0,x_0) = Q(s_0)\int_{\mathbb{R}}(u(0,x+y) - u(s,x))\mathrm{d}y \leq 0$.

Therefore, $(\mathcal{G}, D(\mathcal{G}))$ is a Feller generator. Furthermore, define

$$W(\xi,x) := \sup_\tau \boldsymbol{E}^{\xi,x}\big[\int_0^\tau e^{-as}\bar{f}(\xi(s),X(s))\mathrm{d}s + e^{-a\tau}\bar{g}(\xi(t),X(\tau))\big]. \tag{3.6.22}$$

Since the semi-Markov process $\{X(t)\}_{t\geq 0}$ and the Markov process $\{\xi(t),X(t)\}_{t\geq 0}$ have the same filtration and probability measure, we have $W(0,x) = V(x)$ for $x \in \mathbb{R}$. Since $(\mathcal{G}, D(\mathcal{G}))$ is the generator of the Feller process $\{\xi(t),X(t)\}_{t\geq 0}$, we can use Theorem 3.26 to show (1) and (2) and Theorem 3.8 to show (3). ■

Remark 3.52. *In this example, we have not derived an explicit value function for the optimal stopping problem. However, in Chapter 4, we suggest an iterative scheme to find the value function.*

3.7 Explicit solutions

In this section, we apply the results obtained in Section 3.6 to explicitly derive the solution to the following optimal stopping problem: Find τ^* such that

$$V(x) := \boldsymbol{E}^x[e^{-a\tau^*}g(X(\tau^*))] = \sup_\tau \boldsymbol{E}^x\left[e^{-a\tau}g(X(\tau))\right] \text{ for } x \in [0,\infty), \tag{3.7.1}$$

where $g(x) = (c_2 - x)^+ - (c_1 - x)^+$ with $c_1 < c_2 \in \mathbb{R}$ and $\{X\}_{t\geq 0}$ is a process to be described. $g(x)$ can be understood as the straddle option which is the difference of two options.

3.7.1 Reflected Brownian Motion

In this section, let $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ and suppose $\{X(t)\}_{t\geq 0}$ is a reflected Brownian motion reflected at 0 with state space $E = [0,\infty)$ with core

$$\begin{aligned} D(\mathcal{G}_{ref}) :=& \{u \in C_0^2([0,\infty)); D_x u(0) = 0\},\\ \mathcal{G}_{ref}u(x) :=& \frac{1}{2}D_{xx}u(x) \text{ for } x \in [0,\infty). \end{aligned} \tag{3.7.2}$$

Our aim is to find the explicit optimal stopping time of problem (3.7.1) based on Theorem 3.26. The following corollary is a direct consequence.

Proposition 3.53. *The value function V given by* (3.7.1) *is the unique viscosity solution $w \in \mathcal{C}_b([0,\infty))$ associated with $(\mathcal{G}^*_{ref}, D(G^*_{ref}))$ to*

$$\min(aw - \frac{1}{2}D_{xx}w, w - g) = 0, \tag{3.7.3}$$

where $g(x) = (c_1 - x)^+ - (c_2 - x)^+$ for $x \in [0,\infty)$.

Proof. This result directly follows from Theorem 3.26 by setting $f = 0$ and $g(x) = (c_2 - x)^+ - (c_1 - x)^+$ for $x \in [0,\infty)$. ■

In order to find τ^*, we first need to compute V explicitly as shown below.

Corollary 3.54. *Let X be a reflected Brownian motion reflected at 0. Let C be defined by*

$$C := \min\{p > 0; p(e^{\sqrt{2a}} + e^{\sqrt{-2a}}) \geq g(x)\}, \tag{3.7.4}$$

where a is the discount rate. Then, the value function $V = w$, where

$$w(x) := \begin{cases} C(e^{\sqrt{2a}x} + e^{-\sqrt{2a}x}) & \text{for } x \in [c_1, x^*), \\ g(x) & \text{for } x \in [x^*, \infty), \end{cases} \tag{3.7.5}$$

and

$$x^* = \min\{x; C(e^{\sqrt{2a}x} + e^{-\sqrt{2a}x}) = g(x)\}. \tag{3.7.6}$$

Additionally, the optimal stopping time is $\tau^* = \{t \geq 0; X(t) \in [x^*, c_2]\}$.

Proof. Let us show that w defined by (3.7.5) is a viscosity solution. By definition of C in (3.7.4), $w(x) \geq g(x)$ for $x \in [0, x^*)$. Using (3.7.5), we get $w \geq g$. In what follows, we show the viscosity property for different values of x.

Case 1. Assume that $x \in [0, x^*)$. It is clear from (3.7.5) that w is twice differentiable at x and we have

$$aw(x) - \frac{1}{2}D_{xx}w(x) = 0 \text{ for } x \in [0, x^*). \tag{3.7.7}$$

Since $w(x) \geq g(x)$ for $x \in [0,\infty)$, we have

$$\min(aw(x) - \frac{1}{2}D_{xx}w(x), w(x) - g(x)) = 0.$$

Let $\phi \in D(\mathcal{G}^*_{ref})$ such that $\phi - w$ has a maximum (*respectively*, minimum) at x with $\phi(x) - w(x) = 0$. We first show that $D_{xx}\phi(x) \le (\ge) D_{xx}w(x)$. Assume that $x \in (0, x^*)$. Then x is an interior point. Since w is twice differentiable at x and $\phi - w$ has a maximum (*respectively*, minimum) at x, we have $D_{xx}\phi(x) \le (\ge) D_{xx}w(x)$. Assume now that $x = 0$. Since $\phi \in D(\mathcal{G}^*_{ref})$, we have $D\phi(c_1) = 0$. Then by the fact that $Dw(0) = 0$, we have $D(\phi - w)(0) = 0$. Furthermore, since $\phi - w$ has a maximum (*respectively*, minimum) at $x = 0$, it follows that $D_{xx}(\phi - w)(0) \le (\ge) 0$. Therefore,

$$
\begin{aligned}
&\min(a\phi(0) - \frac{1}{2}D_{xx}\phi(0), \phi(0) - g(0)) \\
\ge (\le)\, &\min(a\phi(0) - \frac{1}{2}D_{xx}w(0), \phi(0) - g(0)) \\
=&0.
\end{aligned}
$$

Hence, w satisfies viscosity property at x.

Case 2. Assume that $x = x^*$ Since $w(x^*) = g(x^*)$, the viscosity subsolution property is satisfied. Then, we only need to show the viscosity supersolution property.

Let $\phi \in D(\mathcal{G}^*_{ref})$ such that $\phi - w$ has a maximum at x^* with $\phi(x^*) - w(x^*) = 0$. Define $w_0(x) := C(e^{\sqrt{2a}x} + e^{-\sqrt{2a}x})$. By (3.7.4) and (3.7.5), we have $w_0(x) \ge w(x)$ for all $x \in [0, \infty)$ and $\phi(x^*) = w(x^*) = w_0(x^*)$. It implies that $\phi - w_0$ also has a maximum at x^* by $\phi(x^*) - w_0(x^*) = \phi(x^*) - w(x^*) = 0$. Hence, since $\phi - w_0$ is twice differentiable and x^* is interior point, $D_{xx}(\phi - w_0) \le 0$. Therefore,

$$
\begin{aligned}
&\min\left(a\phi(x^*) - \frac{1}{2}D_{xx}\phi(x^*), \phi(x^*) - g(x^*)\right) \\
\ge &\min\left(aw_0(x^*) - \frac{1}{2}D_{xx}w_0(x^*), 0\right) \\
=&0.
\end{aligned}
$$

Then, the viscosity supersolution property is satisfied.

Case 3 Assume that $x > x^*$. Since $w(x) = g(x)$, we only need to show the viscosity supersolution. It can be proved similary with Case 1. The result follows by uniqueness of the viscosity solution by Theorem 3.26.

Moreover, the optimal stopping time can be obtained using Theorem 3.8. ■

In the next section, we consider a standard Brownian motions with jumps at the boundary 0.

3.7.2 Brownian motion with jump at boundary

Let $\{X(t)\}_{t\geq 0}$ be a standard Brownian motion which has nonlocal behavior at 0 and state space $E = [0, \infty)$. Then $\{X(t)\}_{t\geq 0}$ is a Feller process whose core is defined by (see for example Taira [2004])

$$\begin{aligned} D(\mathcal{G}_{jump}) &:= \{u \in \mathcal{C}_0^2([0,\infty)); D_{xx}u(0) = 2\lambda \int_0^\infty (u(y) - u(0))\mathrm{d}F(y)\}, \\ \mathcal{G}_{jump}u(x) &:= \frac{1}{2}D_{xx}u(x) \text{ for } x \in [0,\infty), \end{aligned} \tag{3.7.8}$$

where λ is a positive constant and F is a probability distribution function on $(0, \infty)$. The process stays at zero for a positive length of exponential waiting time with parameter λ and then jump back to a random point in $(0, \infty)$ with a probability defined by the distribution function F. Let V_{jump} be the value function of the optimal stopping problem (3.7.1). Then, we have the following result:

Proposition 3.55. *Suppose there exists a solution such that $u(x) = C_1 e^{-\sqrt{2a}x} + C_2 e^{\sqrt{2a}x}$ for $x \in [0, \infty)$, where $C_1, C_2 \in \mathbb{R}$, satisfying*

1. *$u \geq g$,*
2. *There exists $x^* \in [0, \infty)$ such that $u(x^*) = g(x^*)$,*
3. *g is a viscosity supersolution to*

$$aw(x) - \frac{1}{2}D_{xx}w(x) = 0 \text{ for } x \in (x_*, \infty),$$

4. *$a(C_1 + C_2) = \lambda \int_0^{x^*} u(y)\mathrm{d}F(y) + \lambda \int_{x^*}^\infty g(y)\mathrm{d}F(y) - \lambda u(0)$.*

Then,

$$V_{jump}(x) = u_-(x) := \begin{cases} u(x) & \text{for } x \in [0, x^*), \\ g(x) & \text{for } x \in [x^*, \infty). \end{cases} \tag{3.7.9}$$

Proof. Using Theorem 3.26, we only need to show that u_- is a viscosity solution associated to $(\mathcal{G}^*_{jump}, D(\mathcal{G}^*_{jump}))$. Here we only prove the viscosity supersolution property and subsolution property can be shown similarily. Since $u_- \geq g$, we only need to show that for any $\phi \in D(\mathcal{G}^*_{jump})$ such that $\phi \leq u_-$ and $\phi(x_0) = u_-(x_0)$, we have

$$a\phi(x_0) - \frac{1}{2}D_{xx}\phi(x_0) \geq 0. \tag{3.7.10}$$

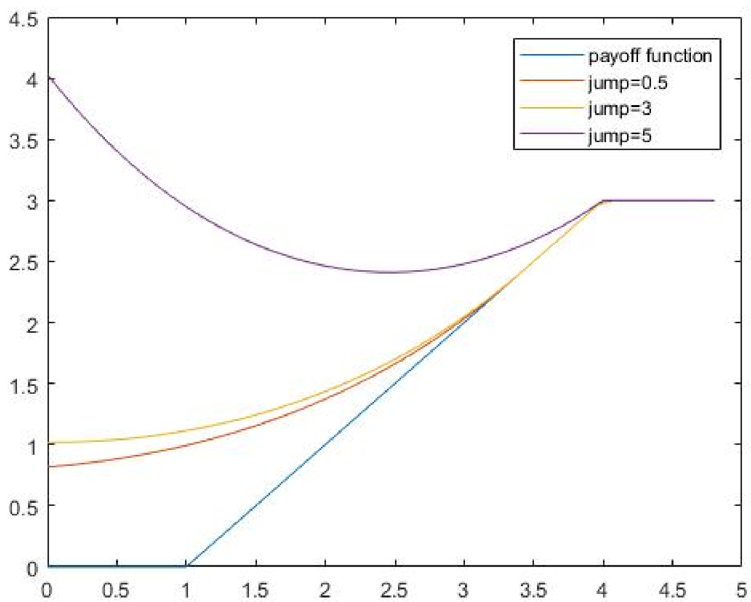
Case 1. Suppose $x_0 \in (x^*, \infty)$. (3.7.10) follows by the condition (3).
Case 2. Suppose $x_0 = x^*$. Since $\phi - u_-$ has a global maximum at x^* by condition (1) and (2) and ϕ and u_- are twice differentiable at x_0, we have $D_{xx}\phi(x_0) \leq D_{xx}u_-(x_0)$.
Case 3. Suppose $x_0 \in (0, x^*)$. Since $u(x) = u_-(x)$ for all $x \in (0, x^*)$ and $au - \frac{1}{2}D_{xx}u = 0$, (3.7.10) holds.
Case 4. Suppose $x_0 = 0$. By the definition of $D(\mathcal{G}^*_{jump})$, we have

$$\begin{aligned} a\phi(0) - \frac{1}{2}D_{xx}\phi(0) &= a\phi(0) - \lambda \int_0^\infty \phi(x)\mathrm{d}F(x) + \lambda\phi(0) \\ &\leq au_-(0) - \lambda \int_0^\infty u_-(x)\mathrm{d}F(x) + \lambda u_-(0) \\ &= 0, \end{aligned}$$

where the fist inequality follows from $u_- \geq \phi$ and $u_-(0) = \phi(0)$ and the last equality follows by condition (4). Hence, (3.7.10) holds when $x_0 = 0$. Therefore, we can conclude u_- is a viscosity supersolution. For the case of the viscosity subsolution, it can be shown similarily. ■

The following figure shows the evolution of the value function with fixed jump size at boundary..

Figure 3.1: Value functions against initial state



In Figure 3.1, we assume that the jump size is fixed at 0.5 (respectively 3 and 5) and the parameter $\lambda = 1$. The graph shows that the value function and exercise point increases with the jump size. We can also mention that the construction of the value function by the viscosity solution can generally be used under weaker condition as compared to the smooth fit principle. Since g is not differential, the smooth fit principle may failed for example if the jump size is equal to 5.

3.7.3 Regime switching boundary

In order to construct a regime switching boundary Feller diffusion, we first construct a regime switching Feller process. Let $\mathcal{S} := \{1, 2, \ldots, N\}$ be a finite discrete space, where N is a positive integer. Let $(\mathcal{A}_i, D(\mathcal{A}_i))$ be the infinitesimal generators of some Feller semigroups on $\mathcal{C}_0(\mathsf{E})$. Then, define the operator $(\mathcal{A}, D(\mathcal{A}))$ as follows:

$$\begin{aligned} D(\mathcal{A}_{regime}) &:= \{u \in \mathcal{C}_0(S \times \mathsf{E}); u(i, \cdot) \in D(\mathcal{G}_i)\}, \\ \mathcal{A}_{regime} u(i, x) &:= \mathcal{A}_i u_i(x) \text{ for } i \in S \text{ and } x \in \mathsf{E}, \end{aligned} \tag{3.7.11}$$

where $u_i(x) := u(i, x)$. By Hille-Yosida theorem, the above generator is the infinitesimal generator of some Feller semigroup. In addition, define the bounded operator

$$\mathcal{F}_{regime} u(i, x) := \sum_{j \in N} q_{ij}(x)(u(j, x) - u(i, x)), \tag{3.7.12}$$

where $q_{ij} \in \mathcal{C}_b(\mathsf{E})$ and $q_{ij} \geq 0$. Since $\mathcal{F}_{regime}$ satisfies the positive maximum principle and $\mathcal{F}_{regime} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$, the operator $((\mathcal{A}_{regime} + \mathcal{F}_{regime}, D(\mathcal{A}_{regime})))$ is the infinitesimal generator of some Feller semigroup.

Next, we construct a regime switching boundary Feller diffusion, that is, the boundary condition is affected by a Markov chain $\{Z(t)\}_{t \geq 0}$ with the state space $\{1, 2\}$. The intensity matrix of the chain is given by

$$\begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix},$$

where $q_1, q_2 > 0$. Let $\{Z(t), X(t)\}_{t \geq 0}$ be a Feller process on the state space $\{1, 2\} \times [0, \infty)$. $\{X(t)\}_{t \geq 0}$ is a one-side diffusion which behaves like Brownian motion in $(0, \infty)$ but is modulated at 0. More precisely, when $X(t)$ touches 0, it either become a sticky Brownian motion or reflected Brownian motion. We denote by $Z(t) = 1$ the state for sticky Brownian motion and $Z(t) = 2$ the tate for reflected Brownian motion. Its infinitesimal generator $(\mathcal{G}, D(\mathcal{G}))$ is defined by:

$$\begin{aligned} D(\mathcal{G}) &:= \{u \in \mathcal{C}_0(\{1,2\} \times [0,\infty); u_i \in \mathcal{C}^2([0,\infty)) \text{ for } i = 1,2, \\ &\qquad D_{xx}u(1,0) = 0 \text{ and } D_x u(2,0) = 0\}, \\ \mathcal{G}u(i,x) &:= \frac{1}{2}D_{xx}u(i,x) + q_i u(3-i,x) - q_i u(i,x) \text{ for } i = 1,2 \text{ and } x \in [0,\infty), \end{aligned} \tag{3.7.13}$$

where $u_i(x) = u(i,x)$ for all $(i,x) \in \{1,2\} \times [0,\infty)$. As a consequence of Theorem 3.26, we have the following characterisation of the value function of the optimal stopping problem (3.7.1):

Corollary 3.56. *There exists a unique pair of viscosity solution $V_1, V_2 \in \mathcal{C}_b([0,\infty))$ such that V_1 is a viscosity solution associated with $(\mathcal{G}_1, D(\mathcal{G}_1))$ to*

$$\min((a+q_1)w - \mathcal{G}_1 w - q_1 V_2, w - g(1,\cdot)) = 0,$$

and V_2 is a viscosity solution associated with $(\mathcal{G}_2, D(\mathcal{G}_2))$ to

$$\min((a+q_2)w - \mathcal{G}_2 w - q_2 V_1, w - g(2,\cdot)) = 0.$$

In order to derive explicit value function, we define several fundamental solutions for optimal stopping problem problems. Let

$$u_k(i,x) = \alpha_{ik} e^{\beta_k x} \tag{3.7.14}$$

$$v_j(i,x) := \begin{cases} q_i \mathcal{R}^{(j)}_{a+q_i} g_{3-i}(x) \text{ for } i = j \\ g_i(x) \text{ for } i \neq j, \end{cases} \tag{3.7.15}$$

$$w_{j1}(i,x) := \begin{cases} e^{\gamma_j x} \text{ for } i = j \\ 0 \text{ for } i \neq j, \end{cases} \tag{3.7.16}$$

$$w_{j2}(i,x) := \begin{cases} e^{-\gamma_j x} \text{ for } i = j \\ 0 \text{ for } i \neq j, \end{cases} \tag{3.7.17}$$

where $i,j = 1,2$, $k = 1,2,3,4$ $\beta_1 = \sqrt{2a}$, $\beta_2 = -\sqrt{2a}$, $\beta_3 = \sqrt{2(a+q_1+q_2}$ and $\beta_4 = -\sqrt{2(a+q_1+q_2)}$, $\alpha_{1k} = 1$ and $\alpha_{2k} = \frac{q_1}{a+q_1-\frac{1}{2}z_j^2}$ and $\gamma_j = \sqrt{2(a+q_j)}$.

Lemma 3.57. *The following hold:*

1. *For any $A_j \in \mathbb{R}$, $u := \sum_{j=1}^4 A_j u_j$ is a solution to*

$$au(i,x) - \mathcal{G}^* u(i,x) = 0 \textit{ for } (i,x) \in \{1,2\} \times (0,\infty) \tag{3.7.18}$$

2. *For any* $B_k \in \mathbb{R}$, $w := \sum_{k=1}^{2} B_k w_{jk} + v_j$ *is a solution to*

$$aw(i,x) - \mathcal{G}^* w(i,x) = 0 \text{ for } (i,x) \in \{j\} \times (0,\infty), \tag{3.7.19}$$

for $j = 1, 2$.

Proof. The result simply follows from direct computations given the parameters. ■ The subsequent result can be seen as a verification theorem for the value function.

Proposition 3.58. *Assume that there exists* $0 \le x_1^* \le x_2^* < \infty$, $A_j \in \mathbb{R}$, $B_k \in \mathbb{R}$ *for* $j = 1,2,3,4$ *and* $k = 1,2$ *such that the function*

$$u(i,x) := \begin{cases} \sum_{j=1}^{4} A_j u_j(i,x) \text{ for } (i,x) \in \{1,2\} \times [0,x_1^*) \\ \sum_{k=1}^{2} B_k w_{jk} + v_j \text{ for } \{1,2\} \times [x_1^*, x_2^*) \\ g(i,x) \text{ for } \{1,2\} \times [x_2^*, \infty), \end{cases} \tag{3.7.20}$$

satisfies

1. $u \ge g$,
2. $u \in \mathcal{C}_b(\{1,2\} \times [0,\infty)$,
3. *u is a viscosity solution to*

$$\min(au(i,x) - \mathcal{G}^* u(i,0), u(i,x) - g(i,x)) = 0 \text{ for } (i,x) \in \{(1,0),(2,0),(j,x_1^*)\} \tag{3.7.21}$$

4. *u is a viscosity supersolution to*

$$au(i,x) - \mathcal{G}^* u(i,x) = 0 \text{ for } (i,x) \in \{j\} \times [x_2^*,\infty) \cup \{3-j\} \times [x_1^*,\infty) \tag{3.7.22}$$

Then, the value function $V = u$.

Proof. To show u is the viscosity solution, we divide the state space into 3 cases,

(i) For $(i,x) \in \{1,2\} \times (0,x_1^*) \cup \{j\} \times (x_1^*, x_2^*)$, the viscosity property is given by Lemma 3.57

(ii) For $(i,x) \in \{(1,0),(2,0),(j,x_1^*)\}$, the viscosity property follows from condition (3)

(iii) For $(i,x) \in \{j\} \times [x_2^*, \infty) \cup \{3-j\} \times [x_1^*, \infty)$, the viscosity property follows from condition (1) and condition (4).

■

Using Proposition 3.58, we need to find A_j, $j = 1,2,3,4, B_k$, $k = 1, 2x_1^*$ and x_2^* such that the viscosity property is satisfied at the following 5 points; $\{(1,0),(2,0),(1,x_1^*),(1,x_2^*),(2,x_1^*)\}$ (respectively $\{(1,0),(2,0),(2,x_1^*),(2,x_2^*),(1,x_1^*)\}$), and the continuity property is satisfied at the following 3 points $\{(1,x_1^*),(1,x_2^*),(2,x_1^*)\}$ (respectively $\{(2,x_1^*),(2,x_2^*),(1,x_1^*)\}$).

We can then derive the explicit expression of the value function as follows:

Corollary 3.59. *Let A_j, B_k, $c_1 < x_1^* < x_2^* \leq c_2$, $l \in \{1,2\}$ such that*

$$\begin{cases} \sum_j A_j u_x(1,0) = 0, \\ \sum_j A_j u_{xx}(2,0) = 0, \\ \sum_j A_j u_j(l,x_1^*) = \sum_k B_k w_{lk}(1,x_1^*) + v_l(x_1^*), \\ \sum_j A_j u_j(3-l,x_1^*) = g(3-l,x_1^*), \\ \sum_k B_k w_{lk}(,x_2^*) + v_{3-l}(x_2^*) = g(l,x_2^*), \\ \sum_j A_j (u_j)_x(l,x_1^*) = \sum_k B_k (w_{lk})_x(1,x_1^*) + (v_l)_x(x_1^*) \end{cases} \tag{3.7.23}$$

and $\sum_j A_j u_j - g$ has a local minimum at $(3-l, x_1^*)$ and $\sum_j B_k u_k + v_l - g$ has a local minimum at (l, x_2^*). If $u \geq g$, then u is the value function.

For fixed numerical values of c_1, c_2, q_1, q_2, and a, we show in the next example that we can find the above parameters A_j, $j = 1,2,3,4, B_k$, $k = 1, 2x_1^*$ and x_2^* and thus derive the value function.

Assume that $a = 0.1$. Figure 3.2 depicts the evolution of the value functions

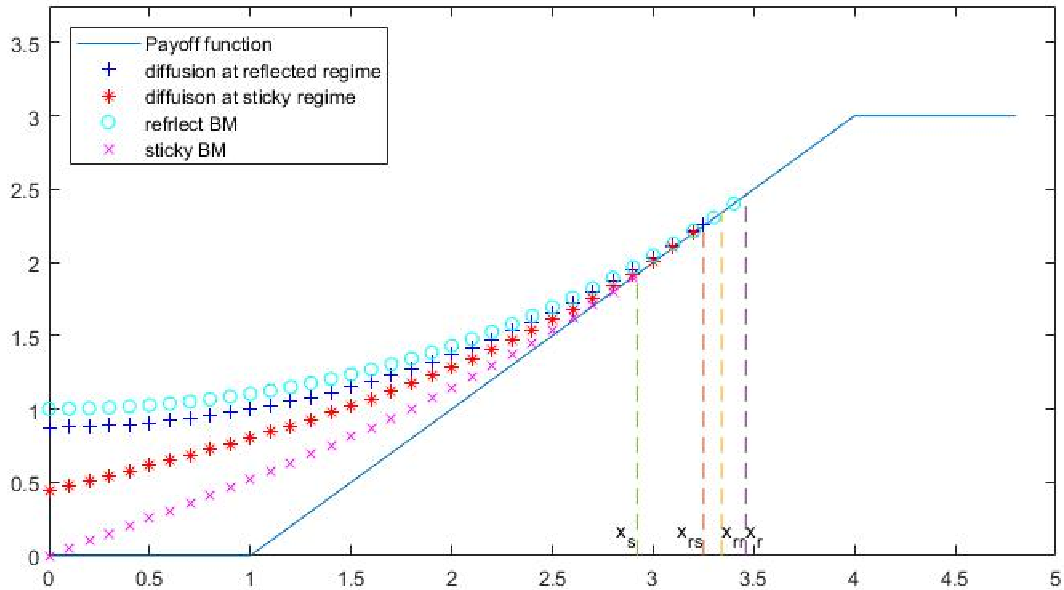
$$V(x) = \sup_\tau \boldsymbol{E}^{i,x}\left[e^{-a\tau}((X(t)-c1)^+ - (X(t)-c_2)^+)\right] \tag{3.7.24}$$

where $c_1 = 1$ and $c_2 = 4$ for regime switching diffusion with reflected boundary and sticky boundary with the intensity matrix

$$\begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

where state 1 represents the reflected boundary and state 2 represents the sticky boundary. x_s, x_{rs}, x_{rr} and x_r are the exercise points in the cases of sticky Brownian motion, diffusion at sticky regime, diffusion at reflected regime and reflected Brownian motion, respectively. Figure 3.2 depicts the evolution of the value function of reflected Brownian motion and sticky Brownian motion with our regime switching respectively. The sticky Brownian motion has an absorbing point at 0

Figure 3.2: Value functions against initial state



and the payoff function at 0 equals 0. This means that the value function of the optimal stopping problem for sticky Brownian motion at 0 is 0 which is smaller than the value function for reflected Brownian motion at 0. Therefore, the exercise points x_r for reflected Brownian motion is larger than that of the sticky Brownian motion x_s. The graph also shows that the value function of this regime switching process will stay between the above two value functions. This is in line with the intuition. Additionally, the graph shows that the exercise points x_{rs} and x_{rr} are between x_s and x_r.

Chapter 4

Iterative optimal stopping methods

In this chapter, we study an iterative optimal stopping method for optimal stopping problem for general Feller process. This approach gives a numerical method to approximate the value function and suggest a way of finding the unique viscosity solution associated to the optimal stopping problem. The literature on this topic was covered in Section 1.3. We apply our result to study several optimal stopping problem. In particular, the method enable us to: reduce the regime switching optimal stopping problem to an iterative optimal stopping problem without regime switching; reduce the optimal stopping problem for semi-Markov process to an iterative optimal stopping problems for two dimensional deterministic process; study an impulse control problem and explicit solutions of one dimensional regular Feller diffusion and study an optimal stopping problem of random discount which can be zero.

4.1 Problem formulation

We first formulate the problem we wish to solve. We start by defining the operator $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ by

$$\mathcal{T}_{\mathsf{F},\mathsf{G}}w := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^\tau e^{-as}\mathsf{F}w(X(s))\mathrm{d}s + e^{-a\tau}\mathsf{G}w(X(\tau)) \right], \tag{4.1.1}$$

where $X(s)$ is a Feller process with state space E and $a > 0$ is constant discount rate, $\mathsf{F} : B(\mathsf{E}) \to B(\mathsf{E})$ and $\mathsf{G} : B(\mathsf{E}) \to B(\mathsf{E})$. In this chapter, we consider the following dynamic programming equation

$$w = \mathcal{T}_{\mathsf{F},\mathsf{G}}w. \tag{4.1.2}$$

Let us remark that the above problem can be thought of as an impulse control problem (see Section 4.3), when G is of the form

$$\mathsf{G}u(x) = \sup_{y\in\mathsf{E}}(u(y) + K(x,y)) \text{ for } u \in B(\mathsf{E}),$$

with $K : \mathsf{E}\times\mathsf{E} \to \mathbb{R}$. On the contrary, Section 4.4 considers an optimal stopping problem for the stochastic process constructed by the perturbation, where F is a perturbation operator. Finally, Section 4.5 works with an optimal stopping problem with a non-negative random discount rate which can be 0.

In general, we aim at showing that under certain conditions, the solution to (4.1.2) is the unique viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\min(aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w) = 0 \tag{4.1.3}$$

Recall Definition 3.1 in Chapter 3 for the viscosity solution to

$$\min(aw - \mathcal{A}w - f, w - g) = 0.$$

We present the definition of viscosity subsolutions (*respectively*, supersolutions) to (4.1.3) as follows.

Definition 4.1. *A function $w \in USC(\mathsf{E})$ (respectively, $w \in LSC(\mathsf{E})$) is a viscosity subsolution (respectively, supersolution) associated with $(\mathcal{A}, D(\mathcal{A}))$ to (4.1.3) if for all $\phi \in D(\mathcal{A})$ such that $\phi - w$ has a global minimum (respectively, maximum) at $x_0 \in \mathsf{E}$ with $\phi(x_0) = w(x_0)$,*

$$\min(a\phi(x_0) - \mathcal{A}\phi(x_0) - \mathsf{F}w(x_0), \phi(x_0) - \mathsf{G}w(x_0)) \le (\ge) 0. \tag{4.1.4}$$

Furthermore, $w \in \mathcal{C}(\mathsf{E})$ is a viscosity solution associated with $(\mathcal{A}, D(\mathcal{A}))$ to (4.2.16) if it is both a viscosity supersolution and a viscosity subsolution.

Let us now present the main theorems of this work. We first show that there exists a unique solution to (4.1.2) (see Theorem 4.6). Then,Theorem 4.8 states that the solution to (4.2.6) can be characterised by the unique viscosity solution to (4.1.2). In the end, we will introduce a numerical method to derive the solution to (4.1.2) by an iterative scheme of optimal stopping problems with its speed of convergence.

4.2 Main theorems

4.2.1 Dynamic programming equation $w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w$

We first start with this section by recalling some basic definitions.

Definition 4.2. *Let $\mathcal{Z}$ be an operator mapping from $\mathcal{C}_b(\mathsf{E})$ to itself.*

1. *$\mathcal{Z}$ is monotonic if for any $u_1 \geq u_2$, $\mathcal{Z}u_1 \geq \mathcal{Z}u_2$.*
2. *$\mathcal{Z}$ is convex if $u_1, u_2 \in \mathcal{C}_b(\mathsf{E})$ and $0 \leq p \leq 1$, then $\mathcal{Z}(pu_1 + (1-p)u_2) \leq p\mathcal{Z}u_1 + (1-p)\mathcal{Z}u_2$.*

We make the following standard assumptions for the operators of F and G.

Assumption 2.

1. *$\mathsf{F} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$ and $\mathsf{G} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.*
2. *The operators F and G are monotonic and convex.*

As a direct consequence of the above assumption, we have the following result.

Lemma 4.3. *Suppose that Assumption 2 holds. Then,*

1. *$\mathcal{T}_{\mathsf{F},\mathsf{G}} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$,*
2. *$\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is monotonic and convex.*

Proof. (1) Let $u \in \mathcal{C}_b(\mathsf{E})$, $f_u := \mathsf{F}u$ and $g_u := \mathsf{G}u$. By (1) in Assumption 2, $f_u, g_u \in \mathcal{C}_b(\mathsf{E})$. Therefore, using Theorem 3.3, the value function of the optimal stopping problem is in $\mathcal{C}_b(\mathsf{E})$.

(2) The operator $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is monotonic and convex follows directly from the fact that the operators F and G are also monotonic and convex. ∎

On the other hand, we also make the following assumption.

Assumption 3.

1. *There exists a positive constant $\kappa > 0$ and $w_+ \in \mathcal{C}_b(\mathsf{E})$ such that*

$$w_+(x) - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+. \tag{4.2.1}$$

2. *There exists $p_1, p_2 \in \mathbb{R}$ satisfying $0 \leq p_1 \leq a$, $0 \leq p_2 \leq 1$ and $\min(p_1/a, p_2) < 1$ such that*

$$\mathsf{F}(u + C) - \mathsf{F}u \leq p_1 C \text{ and } \mathsf{G}(u + C) - \mathsf{G}w \leq p_2 C \tag{4.2.2}$$

for all $u \in \mathcal{C}_b(\mathsf{E})$ and constant function $C > 0$.

Remark 4.4. *Before proving, we first mention that Assumption 2 is necessary to make sure that the solution to $w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w$ is continuous and unique, whereas Assumption 3 provides the upper and lower bounded for that solution. We will see these in more detail in what follows.*

The next lemma will be needed in the proof of our results.

Lemma 4.5. *Suppose Assumption 3 holds.*

1. *Let $\kappa > 0$ and $w_+ \in \mathcal{C}_b(\mathsf{E})$ satisfying (4.2.1). Then, for any constant function $c > 0$, we have*

$$(w_+ + c) - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}}(w_+ + c). \tag{4.2.3}$$

2. *There exists a function $w_0 \in \mathcal{C}_*(\mathsf{E})$ such that*

$$w_0 \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0. \tag{4.2.4}$$

Proof. (1) For any $c > 0$, using (2) in Assumption 3, we have

$$\begin{aligned}
\mathcal{T}_{\mathsf{F},\mathsf{G}}(w_+ + c) &= \sup_\tau \boldsymbol{E}^x \left[\int_0^\tau e^{-as} \mathsf{F}(w_+ + c)(X(s)) \mathrm{d}s + e^{-a\tau} \mathsf{G}(w_+ + c)(X(\tau)) \right] \\
&\leq \sup_\tau \boldsymbol{E}^x \left[\int_0^\tau e^{-as} \big(\mathsf{F} w_+(X(s)) + ac \big) \mathrm{d}s + e^{-a\tau} \big(\mathsf{G} w_+(X(\tau)) + c \big) \right] \\
&= \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+ + c \\
&\leq w_+ + c - \kappa,
\end{aligned}$$

where the first inequality is from (2) in Assumption 3, (4.2.2) and the last inequality is from (1) in Assumption 3

(2) We first assume that $p_1/a < 1$. Let M be a consant such that

$$M \geq \|F(0)\|_\infty / (a - p_1).$$

Define a constant function ϕ by $\phi(x) := -M$ for all $x \in \mathsf{E}$. Then $a\phi - \mathsf{F}\phi \leq 0$. Let $\phi_0(x) := 0$ for all $x \in \mathsf{E}$. In fact, by (2) in Assumption 3, $\mathsf{F}\phi_0 - \mathsf{F}\phi \leq p_1 M$ and thus $-\mathsf{F}\phi \leq p_1 M - \mathsf{F}\phi_0$. Hence, $a\phi - \mathsf{F}\phi \leq (a - p_1)\phi - \mathsf{F}\phi_0 \leq 0$.

Since $\mathcal{A}\phi \geq 0$ by the positive maximum principle, $a\phi - \mathcal{L}\phi - \mathsf{F}\phi \leq 0$ and then

$$\min(a\phi - \mathcal{A}\phi - \mathsf{F}\phi, \phi - \mathsf{G}\phi) \leq 0, \tag{4.2.5}$$

that is, ϕ is a viscosity subsolution to (4.2.5). On the other hand, since $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi$ is the value function for optimal stopping problem, $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi$ and then ϕ are the viscosity solution to (4.2.5) (See Theorem 3.26). By the comparison principle (see Theorem 3.27), we have $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi \geq \phi$. The case when $p_2 < 1$ can be proved similarly. ■

Using this lemma, we present an existence and uniqueness result of the dynamic programming equation.

Theorem 4.6. *Suppose that Assumption 2 and Assumption 3 holds. There exists a unique solution* $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w. \tag{4.2.6}$$

Proof. Using (2) in Lemma 4.5, there exist $w_0 \in \mathcal{C}_b(\mathsf{E})$ such that

$$\mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \geq w_0.$$

Define $w_{n+1} := \mathcal{T}_{\mathsf{F},\mathsf{G}} w_n$ for $n \in \mathbb{N}$. By (1) in Assumption 3, there exists $\kappa > 0$, $w_+ \in \mathcal{C}_b(\mathsf{E})$ such that

$$w_+ - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+.$$

Since $w_0 \in \mathcal{C}_b(\mathsf{E})$, we have $w_1 = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \in \mathcal{C}_b(\mathsf{E})$. There exists $c_0 > 0$ such that $w_1 < c_0$. Choose c large enough and define $w_+^* := w_+ + c \geq w_1$. Then, by (1) in Lemma 4.5, we have

$$w_+^* - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^*. \tag{4.2.7}$$

Thus, we obtain

$$0 \leq w_1 - w_0 \leq w_+^* - w_0.$$

Now, we want to prove that there exists $0 \leq \gamma < 1$ such that

$$w_{n+1} - w_n \leq \gamma^n (w_+^* - w_n) \text{ for all } n \in \mathbb{N}. \tag{4.2.8}$$

We prove this by induction. (4.2.8) holds when $n = 0$, assume that (4.2.8) holds for all $n \leq m$ where m is some positive integer. We want to prove that

$$w_{m+2} - w_{m+1} \leq \gamma^{m+1}(w_+^* - w_{m+1}). \tag{4.2.9}$$

Since $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is monotonic by Lemma 4.3 and $w_1 = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \geq w_0$, it follows that the sequence $\{w_n\}_{n\in\mathbb{N}}$ is increasing. By(4.2.9), we have

$$w_{m+1} \leq \gamma^m w_+^* + (1 - \gamma^m) w_n.$$

Thus by monotonicity and convexity of $\mathcal{T}_{\mathsf{F},\mathsf{G}}$, we have

$$\begin{aligned}
\mathcal{T}_{\mathsf{F},\mathsf{G}} w_{m+1} &\leq \mathcal{T}_{\mathsf{F},\mathsf{G}}(\gamma^m w_+^* + (1-\gamma^m) w_m) \\
&\leq \gamma^m \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^* + (1-\gamma^m)\mathcal{T}_{\mathsf{F},\mathsf{G}} w_m \\
&\leq \gamma^m (w_+^* - \kappa) + (1-\gamma^m) w_{m+1} \\
&= w_{m+1} + \gamma^m (w_+^* - w_{m+1} - \kappa) \\
&= w_{m+1} + \gamma^m \frac{w_+^* - w_{m+1} - \kappa}{w_+^* - w_{m+1}} (w_+^* - w_{m+1}) \\
&= w_{m+1} + \gamma^m \big(1 - \frac{\kappa}{w_+^* - w_{m+1}}\big)(w_+^* - w_{m+1}) \\
&\leq w_{m+1} + \gamma^m \big(1 - \frac{\kappa}{\|w_+^* - w_0\|_\infty}\big)(w_+^* - w_{m+1}),
\end{aligned}$$

where the last inequality is from the fact that $w_+^* \geq w_m \geq w_0$. Choosing

$$\gamma = \max\left(0, 1 - \frac{\kappa}{\|w_+^* - w_0\|_\infty}\right),$$

we get $w_{n+1} - w_n \leq \gamma^n \|w_+^* - w_n\|_\infty \leq \gamma^n \|w_+^* - w_0\|_\infty$ for all $n \in \mathbb{N}$. Therefore, $\{w_n\}_{n\in N}$ is a Cauchy sequence in $(\mathcal{C}_b(\mathsf{E}), \|\cdot\|_\infty)$ and there exists $w_\infty \in \mathcal{C}_b(\mathsf{E})$ such that w_n uniformly converges to w_∞ and satisfies $\mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty = w_\infty$. The existence of the solution to (4.2.6) is proved.

For the uniqueness, we only need to prove that $w_\infty \in \mathcal{C}_b(\mathsf{E})$ is the unique solution to (4.2.6). It can be proved using the comparison principle as shown below. ■

Using similar arguments in the above theorem, we derive the subsequent comparison principle.

Proposition 4.7. *(Comparison Principle) Suppose that Assumption 2 and Assumption 3 holds. Let w be the solution to* (4.2.6). *If $u \geq (\leq)\mathcal{T}_{\mathsf{F},\mathsf{G}} u$, then $u \geq (\leq) w$.*

Proof. Assume that there exists $v_+ \in \mathcal{C}_b(\mathsf{E})$ satisfying $v_+ \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$. Let us prove that $v_+ \geq w_\infty$. Assume by contradiction that there exists some x_0 such that $v_+(x_0) < w_\infty(x_0)$. Then, since $w_+^* \geq w_\infty$, there exists $0 < \gamma \leq 1$ such that

$$w_\infty(x_0) - v_+(x_0) = \gamma(w_+^*(x_0) - v_+(x_0)). \tag{4.2.10}$$

Since w_∞ satisfies $w_\infty = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty$ and $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is convex, we have

$$\begin{aligned} w_\infty(x_0) = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty(x_0) &= \mathcal{T}_{\mathsf{F},\mathsf{G}}(\gamma w_+^* + (1-\gamma)v_+)(x_0) \\ &\le \gamma \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^*(x_0) + (1-\gamma)\mathcal{T}_{\mathsf{F},\mathsf{G}} v_+(x_0) \\ &\le \gamma(w_+^*(x_0) - \kappa) + (1-\gamma)v_+(x_0), \end{aligned}$$

where the last inequality follows from (4.2.7) and $v_+ \ge \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$. Therefore, there exists $\kappa > 0$

$$w_\infty(x_0) - v_+(x_0) \le \gamma(w_+^*(x_0) - v_+(x_0) - \kappa).$$

Since $\gamma > 0$, this contradicts (4.2.10). Then, $v_+ \ge \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$ implies $v_+ \ge w_\infty$.

On the other hand, assume that there exists $v_- \in \mathcal{C}_b(\mathsf{E})$ satisfying $v_- \le \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$. To prove $v_- \le w_\infty$, assume that there exists some x_0 such that $w_+^*(x_0) \ge v_+(x_0) > w_\infty(x_0)$. Then, similarly, there exists $0 < \gamma \le 1$ such that

$$v_-(x_0) - w_\infty(x_0) = \gamma(w_+^*(x_0) - w_\infty(x_0)). \tag{4.2.11}$$

Then, since $v_- \le \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$ we have

$$v_-(x_0) - w_\infty(x_0) < \gamma(w_+^*(x_0) - w_\infty(x_0) - \kappa).$$

This contradicts (4.2.11). Therefore, $v_- \le \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$ implies $v_- \le w_\infty$.

The conclusion if v is a solution to (4.2.6) then $v = w_\infty$. As from the above $v \ge (\le) \mathcal{T}_{\mathsf{F},\mathsf{G}} v$ implies that $v \ge (\le) w_\infty$ and the result follows. ■

4.2.2 Viscosity Solution

We show in this section that under certain conditions, the solution to (4.2.6) is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation

$$\min(aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w) = 0. \tag{4.2.12}$$

We first recall two results Theorem 3.26 and Theorem 3.27 from Chapter 3 when $\mathsf{F}u = f$ and $\mathsf{G} = g$. They will be used to prove the general cases about F and G later. Define the infinitesimal generator of a Feller process by:

$$\begin{aligned} D(\mathcal{A}) &:= \{u \in \mathcal{C}_*(\mathsf{E}); u - u(\partial) \in D(\mathcal{G})\}, \\ \mathcal{A}u &:= \mathcal{G}(u - u(\partial)), \end{aligned} \tag{4.2.13}$$

where $(\mathcal{G}, D(\mathcal{G}))$ is the core of Feller process X.

Theorem 4.8. *Suppose $f, g \in \mathcal{C}_b(\mathsf{E})$ and $a > 0$. Let V be the value function V defined by*

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \right]. \tag{4.2.14}$$

Then, the value function V is the unique viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ associated with $(\mathcal{A}, D(\mathcal{A}))$ to

$$\min(aw - \mathcal{A}w - f, w - g) = 0, \tag{4.2.15}$$

with $(\mathcal{A}, D(\mathcal{A}))$ given by (4.2.13).

Theorem 4.9. *Suppose $a > 0$ and $f, g \in \mathcal{C}_b(\mathsf{E})$. Let $w_1 \in USC(\mathsf{E})$ and $w_2 \in LSC(\mathsf{E})$ are the viscosity subsolution and supersolution to* (4.2.15)*, respectively. If w_1 and w_2 are bounded from above and below, respectively, then, $w_1 \leq w_2$.*

Using the above theorems, one can characterise the solution to $w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w$ in the viscosity sense.

Theorem 4.10. *Suppose that Assumption 2 and Assumption 3 hold. There exists a unique viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ to*

$$\min(aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w) = 0 \tag{4.2.16}$$

Proof. By Theorem 4.8, a function $w \in \mathcal{C}_b(\mathsf{E})$ is a solution to $\mathcal{T}_{\mathsf{F},\mathsf{G}} w = w$ if and if w is a viscosity solution to (4.2.16). Since there exists a unique solution to $\mathcal{T}_{\mathsf{F},\mathsf{G}} w = w$ by Theorem 4.6, this completes the proof. ■

Proposition 4.11. *(Comparison Principle) Suppose that Assumption 2 and Assumption 3 holds. Let $w_1 \in \mathcal{C}_b(\mathsf{E})$ and $w_2 \in \mathcal{C}_b(\mathsf{E})$ be a viscosity subsolution and supersolution to* (4.2.16)*. Then, $w_1 \leq w_2$.*

Proof. Using Theorem 4.9, we know that if w_1 (*respectively*, w_2) is viscosity subsolution (*respectively*, supersolution), then $w_1 \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_1$ (*respectively*, $w_2 \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_2$). Therefore, by Proposition 4.7, we know that $w_1 \leq w_\infty \leq w_2$. ■

Based on Proposition 4.11, we provide a sufficient conclusion for condition (1) in Assumption 3 to hold.

Corollary 4.12. *Assume there exists a positive constant $\kappa > 0$ and a viscosity supersolution $w_+ \in \mathcal{C}_b(\mathsf{E})$ to*

$$\min(aw_+ - \mathcal{A}w_+ - \mathsf{F}_\kappa w_+, w_+ - \mathsf{G}_\kappa w_+) = 0, \tag{4.2.17}$$

where $\mathsf{F}_\kappa w_+ := \mathsf{F}w_+ + a\kappa$ and $\mathsf{G}_\kappa w_+ := \mathsf{G}w_+ + \kappa$. Then, (1) in Assumption 3 holds.

Proof. Since w_+ is the viscosity supersolution, by Proposition 4.11, we have

$$
\begin{aligned}
w_+(x) &\geq \mathfrak{T}_{\mathbb{F}_\kappa, \mathsf{G}_\kappa} w_+(x), \\
&= \sup_\tau \boldsymbol{E}^x[\int_0^\tau e^{-as}(\mathsf{F}w_+(X(s)) + a\kappa)\mathrm{d}s + e^{-as\tau}(\mathsf{G}w_+(X(\tau)) + \kappa)] \\
&= \kappa + \mathfrak{T}_{\mathsf{F},\mathsf{G}} w_+(x).
\end{aligned}
$$

Then, the proof is finished. ■

4.2.3 Numerical Approximation

Besides the characterization of the value function in the viscosity sense, the proof of Theorem 4.6 also straightforward provides an iterative algorithm derive to the solution to (4.2.6).

Theorem 4.13. *(Iteration of optimal stopping problems) Suppose that Assumption 2 and Assumption 3 hold. Let $w_0 \in \mathcal{C}_b(\mathsf{E})$ such that $w_0 \leq \mathfrak{T}_{\mathsf{F},\mathsf{G}} w_0$. Define*

$$
w_{n+1} := \mathfrak{T}_{\mathsf{F},\mathsf{G}} w_n \text{ for } n \in \mathbb{N}. \tag{4.2.18}
$$

Then, $\{w_n\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}_b(\mathsf{E})$ converging uniformly to the unique solution $w \in \mathcal{C}_b(\mathsf{E})$ to (4.2.6). Additionally, the error term e_n converges to zero with

$$
e_n := \|w - w_n\|_\infty \leq C\gamma^n \tag{4.2.19}
$$

with

$$
\gamma = 1 - \frac{\kappa}{max(c_0, \kappa)} \text{ and } C = c_0^2. \tag{4.2.20}
$$

where $w_+ \in \mathcal{C}_b(\mathsf{E})$ and $\kappa > 0$ satisfies (4.2.1) and $c_0 := \sup_{x\in\mathsf{E}}(w_+(x) - w_0(x))$.

Proof. It can be shown similarly from the proof of Theorem 4.6. Since

$$
w_{n+1} - w_n \leq \gamma^n \|w_+^* - w_0\|_\infty,
$$

we have

$$\begin{aligned}
w - w_n &= \lim_{m\to\infty}(w_{m+n} - w_n) \\
&= \lim_{m\to\infty}\sum_{i=1}^{m}(w_{n+i} - w_{n+i-1}) \\
&\le \lim_{m\to\infty}\sum_{i=1}^{m}\gamma^{n+i-1}\|w_+^* - w_0\|_\infty \\
&= \frac{\gamma^n}{1-\gamma}\|w_+^* - w_0\|_\infty.
\end{aligned}$$

■

Furthermore, since the explicit solution the value function $\mathcal{T}_{\mathsf{F},\mathsf{G}}w$ in each step cannot be found in general, it is also necessary to study the convergence of the scheme if the approximated solutions are used in each step. Let $\mathcal{T}^{(M)}_{\mathsf{F},\mathsf{G}}$ be an approximated operator satisfying the following property:

$$\|\mathcal{T}^{(M)}_{\mathsf{F},\mathsf{G}}u - \mathcal{T}_{\mathsf{F},\mathsf{G}}u\|_\infty \le \frac{\|u\|_\infty}{M}. \tag{4.2.21}$$

Let $w_0^{(M)} := w_0$ and define

$$w_{n+1}^{(M)} := \mathcal{T}^{(M)}_{\mathsf{F},\mathsf{G}}w_n^{(M)} \text{ for } n \in \mathbb{N}. \tag{4.2.22}$$

Proposition 4.14. *Assume that the operator $\mathcal{T}^{(M)}_{\mathsf{F},\mathsf{G}} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$ satisfies for any $u \in \mathcal{C}_b(\mathsf{E})$*

$$0 \le \mathcal{T}_{\mathsf{F},\mathsf{G}}u - \mathcal{T}^{(M)}_{\mathsf{F},\mathsf{G}}u \le \frac{\|u\|_\infty}{M}. \tag{4.2.23}$$

Then the error term $e_n^{(M)}$ is

$$e_n^{(M)} := \|w_n^{(M)} - w\|_\infty \le \frac{n}{M}C_0 + C\gamma^n. \tag{4.2.24}$$

where γ and C is defined in (4.2.20) *and $C_0 := \max(\|w_0\|_\infty, \|w_+\|_\infty)$.*

Proof. We first prove that for any $n \in \mathbb{N}$

$$0 \le w_n - w_n^{(M)} \le \frac{n}{M}C \tag{4.2.25}$$

$$w_0 - \frac{C}{M} \le w_n^{(M)} \le w_n \le w_+ \tag{4.2.26}$$

where $C_0 := \max(\|w_0\|_\infty, \|w_+\|_\infty)$. When $n = 1$, (4.2.25) and (4.2.26) hold. By iterative, we assume that (4.2.25) and (4.2.26) hold when $n = m$. We will prove that (4.2.25) and (4.2.26) also hold when $n = m + 1$ as follows.

$$\begin{aligned}
w_{m+1} - w_{m+1}^{(M)} &= \mathcal{T}_{\mathsf{F},\mathsf{G}} w_m - \mathcal{T}_{\mathsf{F},\mathsf{G}}^{(M)} w_m^M \\
&= (\mathcal{T}_{\mathsf{F},\mathsf{G}} w_m - \mathcal{T}_{\mathsf{F},\mathsf{G}} w_m^{(M)}) + (\mathcal{T}_{\mathsf{F},\mathsf{G}} w_m^{(M)} - \mathcal{T}_{\mathsf{F},\mathsf{G}}^{(M)} w_m^{(M)}) \\
&\le \|w_m - w_m^{(M)}\|_\infty + \frac{\|w_m^{(M)}\|_\infty}{M} \\
&\le \frac{m+1}{M} C_0.
\end{aligned}$$

■

The form of the viscosity solution to (4.2.16) can usually be founded in stochastic control problems. Examples of such problems satisfying Assumption 2 and Assumption 3 are introduced in what follows.

4.3 Impulse control

In this section, we show the link between the value function of some impulse control problem and the viscosity solution to some HJB equations. Such relationship has been studied before (see for example Guo and Wu [2009]; Øksendal and Sulem [2007]; Seydel [2009] and Robin [1978] for general Markov processes). In this section, we extend the above results in two directions. First, we characterise the value function of an impulse control for Feller processes as a viscosity solution to an HJB equations; second, we relax the assumption of the performance functional (see (3) in Assumption 4). The later condition can be derived from (1) in Assumption 3, which can be found in Seydel [2009]. Note however that Seydel [2009] studies impulse control problem for jumping diffusions and use an approach different to the iterative approach for general Feller processes utilised in this work.

Consider a general Feller Markov process and let us introduce the following impulse control problem studied in Robin [1978]. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, \boldsymbol{P}^x)$ be a Markov process. Define $\Omega_\infty := (\Omega)^{\times\infty}$ and $\mathcal{F}_t^n := \mathcal{F}_t^{\times n}$ for $n \in \mathbb{N}$. The shift operator is defined by $\theta_t^n \omega(s) := (\theta_t \omega_1(s), \theta_t \omega_2(s), \ldots, \theta_t \omega_n(s))$ for $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_\infty$. A sequence of $\pi := \{\tau_i, \xi_i\}_{i\in\mathbb{N}}$ is called an *admissible control* strategy if

1. τ_i is a $\mathcal{F}_t^n \times \{\emptyset, \Omega\}^{\times\infty}$-measurable stopping time, $\tau_i \le \tau_{i+1}$ and $\lim_n \tau_n = \infty$.
2. ξ_i is $\mathcal{F}_{\tau_i} \times \{\emptyset, \Omega\}^{\times\infty}$-measurable.

The trajectory of the controlled process $\{X^\pi(t)\}_{t\ge 0}$ is defined by using coordinates $X_t(\omega) = X_t(\omega_n)$ for $t \in [\tau_n, \tau_n + 1)$ and $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_\infty$. The process X^π

shifts to a new state ξ_n at τ_n and it generates a new probability measure $\boldsymbol{P}^{\pi,x}$ (see for example [Robin, 1978, Section 5] for more information). The impulse control problem is to find the optimal admissible strategy π that maximizes

$$J(x,\pi) := \boldsymbol{E}^{\pi,x}\left[\int_0^\infty e^{-as} f(X^\pi(s))ds + \sum_{i=1}^\infty e^{-a\tau_i} K(X^\pi(\tau_i^-)), X^\pi(\tau_i))\right], \quad (4.3.1)$$

where $f : \mathsf{E} \to \mathbb{R}$ is a continuous bounded function and $K : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$ is the reward obtained at ith impulse control. The value function of the above problem is defined by

$$V(x) := \sup_\pi J(x,\pi). \quad (4.3.2)$$

4.3.1 Main Results

In this section, the notion of viscosity solution is often used to solve the variational inequality associated with the value function for such impulse control problem (see for example Davis et al. [2010]; Guo and Wu [2009]; Seydel [2009]). In the above case, the value function can be characterized by the viscosity solution to

$$\min(aw - \mathcal{A}w - f, g - \mathcal{M}w) = 0, \quad (4.3.3)$$

with

$$\mathcal{M}u(x) := sup_{y\in\mathsf{E}}(u(y) + K(x,y)). \quad (4.3.4)$$

In order to solve the underlying problem, we make the following assumption such that Assumption 2 and Assumption 3 are satisfied.

Assumption 4.

1. $\mathcal{M} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.
2. *There exists a constant* $C > 0$ *such that*

$$K(x,y) + K(y,z) \leq K(x,z) - C \textit{ for all } x,y,z \in \mathsf{E}. \quad (4.3.5)$$

3. *Fix the constant* $C > 0$ *from (2). There exists a function* $u \in \mathcal{C}_b(\mathsf{E}) \cap \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$*, a point* $z_0 \in \mathsf{E}$ *and a constant* $\kappa > 0$ *such that for all* $x \in \mathsf{E}$*,*

$$0 \leq u(x) - K_{z_0}(x) \leq C - \kappa \quad (4.3.6)$$

where $K_{z_0}(x) := K(x, z_0)$*.*

Remark 4.15. *(1) and (2) in Assumption 4 are common in the literature to impulse control problems. In general, when studying general impulse control problem, most papers (see for example Davis et al. [2010]; Guo and Wu [2009]) use the following stronger assumption in the place of (3) in Assumption 4: $K(x, y) < -C$ for all $x, y \in \mathsf{E}$. However, the above assumption failed to be satisfied in some interesting applications in finance. Thus, (3) in Assumption 4 enables us to consider more general example of application.*

Therefore, under the aforementioned assumption, the problem (4.3.2) has a solution which is presented in the proposition below. Notice that this is a direct consequence of Theorem 4.8.

Proposition 4.16. *Suppose that Assumption 4 holds and $f \in \mathcal{C}_b(\mathsf{E})$.*

1. *There exists a unique viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ to*

$$\min(aw - \mathcal{A}w - f, w - \mathcal{M}w(x)) = 0. \tag{4.3.7}$$

2. *Additionally, suppose that the value function $V \in \mathcal{C}_b(\mathsf{E})$ defined by (4.3.2) satisfies the following dynamic programming equation*

$$w(x) = \mathcal{T}_{f,\mathcal{M}}w := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-as} f(X(s))ds + e^{-a\tau}\mathcal{M}w(X(\tau)) \right]. \tag{4.3.8}$$

   Then, $V = w$, where w is the unique viscosity solution to (4.3.7).

Proof. (1) Denote $\mathsf{F}u := f$ and $\mathsf{G}u := \mathcal{M}u$ for all $u \in \mathcal{C}_b(\mathsf{E})$. Then, Assumption 2 follows from (1) in Assumption 4 and the convexity and monotonicity properties of G can be proved as Guo and Wu [2009]. Additionally, the convexity and monotonicity property of F follows from the fact that $f \in \mathcal{C}_b(\mathsf{E})$. Furthermore, since $\mathcal{M}(u+c) = \mathcal{M}u+c$ for any $u \in \mathcal{C}_b(\mathsf{E})$ and constant function c, we only need to verify (1) in Assumption 3. Let us fix $z_0 \in \mathsf{E}$. Using (2) in Assumption 4, there exists a constant $C > 0$ such that

$$K(x, y) + K(y, z_0) \leq K(x, z_0) - C.$$

Then, there exists $u \in \mathcal{C}_b(\mathsf{E}) \cap \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$ such that for any $x \in \mathsf{E}$,

$$\begin{aligned}
&u(x) - \sup_{y\in\mathsf{E}}(u(y) + K(x, y)) \\
\geq &u(x) - \sup_{y\in\mathsf{E}}(u(y) + K(x, z_0) - K(y, z_0)) + C \\
\geq &u(x) - K(x, z_0) - \sup_{y\in\mathsf{E}}(u(y) - K(y, z_0)) + C \\
\geq &0 - (C - \kappa) + C \geq \kappa,
\end{aligned}$$

where the second inequality is from (3) in Assumption 4. Hence, $u - \mathcal{M}u \geq \kappa$.

Furthermore, since $u \in \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$, there exists $h \in \mathcal{C}_b(\mathsf{E})$ such that $h = (a - \mathcal{A})u$. Define $u^* := u + (\|h\|_\infty + \|f\|_\infty)/a + \kappa$. We have $(a - \mathcal{A})u^* = h + (\|h\|_\infty + \|f\|_\infty) + a\kappa \geq f + a\kappa$. Additionally, Since $u - \mathcal{M}u \geq \kappa$ implies $u^* - \mathcal{M}u^* \geq \kappa$, u^* satisfies

$$(au^* - \mathcal{A}u^* - f - a\kappa, u^* - \mathcal{M}u^* - \kappa) = 0.$$

Then, by Corollary 4.12, (1) in Assumption 3 is shown.

(2) The proof of the claim follows by applying Theorem 4.8. ∎

Furthermore, the value function V defined in (4.3.2) can be solved by the iterative optimal stopping method using Theorem 4.6.

Proposition 4.17. *Suppose that Assumption 4 holds. Let* $w_0 := \mathcal{R}_a f$ *and* $w_{n+1} := \mathcal{T}_{f,\mathcal{M}} w_n$*, where* $\mathcal{T}_{f,\mathcal{M}}$ *is defined by* (4.3.8)*. Then, the sequence of functions* $\{w_n\}_{n\in\mathbb{N}}$ *converges to* w *uniformly as* $n \to \infty$*.*

Proof. Since w_0 is the subsolution to

$$\min(aw - \mathcal{A}w - f, w - \mathcal{M}w) = 0,$$

then $w_0 \leq \mathcal{T}_{f,\mathcal{M}} w_0$. The claim follows from Theorem 4.6 directly. ∎

Using these more general results in Proposition 4.16 and Proposition 4.17, we examine two impulse control problems in the following sections. In the first example, we study impulse control for general Markov process whereas the one-dimensional regular Feller diffusion is considered in the second example.

4.3.2 Equivalence between the optimal stopping problems and impulse control problems

Initially, we show that one specific kind of impulse control problems can be considered under the optimal stopping framework, where the premium function K in the impulse control problem defined by (4.3.2) takes the following form

$$K(x,y) := \begin{cases} k(x) - k(y) - c(x) & \text{for } (x,y) \in \mathsf{E} \times S, \\ -\infty & \text{for } (x,y) \notin \mathsf{E} \times S, \end{cases} \tag{4.3.9}$$

where S is a non-empty Borel subset of E. The above can be interpreted as follows: the intervention for state from state x to y generates $k(x) - k(y)$ but the cost $k(x)$ depends only on the state before the impulse. Let the value function be defined by

$$V^{(k)}(x) := \sup_{\pi} \boldsymbol{E}^{\pi,x}\left[\int_0^{\infty} e^{-as} f(X^{\pi}(s))ds + \sum_{i=1}^{\infty} e^{-a\tau_i}\Big(k(X^{\pi}(\tau_i^-) - k(X^{\pi}(\tau_i)) - c(X^{\pi}(\tau_i^-))\Big)\right]. \tag{4.3.10}$$

More specifically, the above choice of K and value function can be found in some practical examples.

Example 4.18.

1. *Dividend and Injection with fixed cost. A popular example pertains to optimal dividend in financial and actuarial mathematics. Set a compact subset* $\mathsf{E} \subseteq \mathbb{R}$, $k(x) = x$ *and* $c(x) = c_0 > 0$ *so that* $K(x, y) = x - y - c_0$. *The associated optimal stopping control problem can be seen as an optimal proportional dividend and capital injection problem with fixed cost* k_1. *Such problem can be seen for the company who pays dividend or invest with only fixed costs.*

2. *Dividend and injection for exponential Lévy process. Suppose* $Y(t) = e^{X(t)}$, *where* $X(t)$ *is a Lévy process. Then impulse control problem of* Y *for dividend and injection for such problem can use the following function* $k(x) = e^x$ *and* $c(x) = c_0$.

It can be seen that the premium function k in the first example coincides with Guan and Liang [2014]. On the other hand, the premium function in (2) is consistent with such problem where the process is geometric BM. For this kind of problems, we show that a generalized method is applicable through deriving viscosity solutions to its corresponding HJB equations. Using Theorem 4.9, we have the following proposition

Proposition 4.19. *Suppose that* $f \in \mathcal{C}_b(\mathsf{E})$, $k \in \mathcal{C}_b(\mathsf{E})$, $c \in \mathcal{C}_b(\mathsf{E})$ *and* $c \geq C_0 > 0$.

1. *The value function* $V^{(k)}$ *is the unique viscosity solution* $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$\min(aw - \mathcal{A}w - f, w - \mathcal{M}_k w) = 0. \tag{4.3.11}$$

where

$$\mathcal{M}_k w(x) := \sup_{y \in S}(w(y) - k(y) + k(x) - c(x)). \tag{4.3.12}$$

2. *There exists a unique pair of constant and function* $(\delta, w_\delta) \in \mathbb{R} \times \mathcal{C}_b(\mathsf{E})$ *such that* w_δ *is a viscosity solution*

$$\min(aw - \mathcal{A}w - f, w - (k - c - \delta)) = 0 \tag{4.3.13}$$

and $\delta = \sup_{y \in S}(w_\delta(y) - k(y))$. *Furthermore, the value function* $V^{(k)} = w_\delta$.

Proof. (1) We only need to verify Assumption 4. By (4.3.12), since

$$\mathcal{M}_k w(x) = \sup_{y \in S}(w(y) - k(y)) + k(x) - c(x)$$

and $k \in \mathcal{C}_b(\mathsf{E})$, then we have $\mathcal{M}_k : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$. Then, (1) in Assumption 4 holds. For (2) in Assumption 4, it can be shown as follows

$$\begin{aligned} K(x, y) + K(y, z) &\leq k(x) - k(y) - c(x) + k(y) - k(z) - c(y) \\ &\leq K(x, z) - C \end{aligned}$$

for $x, y, z \in \mathsf{E}$. Finally, let $z_0 \in S$ and $K_{z_0} := K(x, z_0) = k(x) - k(z_0) - c(x)$. Then, since $K_{z_0} \in \mathcal{C}_b(\mathsf{E})$ by the fact that $k, c \in \mathcal{C}_b(\mathsf{E})$, there exists a constant $0 < \varepsilon < C_0/2$ and $u_\varepsilon \in \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$ such that $\|u_\varepsilon - K_{z_0}\|_\infty \leq \varepsilon$, that is

$$0 \leq u_\varepsilon - K_{z_0} \leq 2\varepsilon.$$

Thus there exists $\kappa = C_0 - 2\varepsilon > 0$ satisfying (3) in Assumption 4.

(2) Since $\mathcal{M}_k w(x) = k(x) - c(x) + \sup_{y \in S}(w(y) - k(y))$ by rewriting (4.3.12), we have

$$\min(aw - \mathcal{A}w - f, w - k + c - \sup_{y \in S}(w(y) - k(y))) = 0. \tag{4.3.14}$$

Hence, there exists a unique (w_δ, δ) in the argument. ■

For the cases where δ cannot be explicitly computed, we demonstrate a numerical method to approximate the value function in the following corollary.

Corollary 4.20. *Let* $u_0(x) = \mathcal{R}_a f$. *Let* $\delta_n := \sup_{y \in S}(u_n(y) - k(y))$, $g_n := k - c - \delta_n$ *and* u_{n+1} *is the unique viscosity solution to*

$$\min(aw - \mathcal{A}w - f, w - g_n) = 0. \tag{4.3.15}$$

Then, u_n *uniformly convergence to* V.

From Corollary 4.20, the pair of solutions (δ, w_δ) does not always have an analytical form. However, in Corollary 4.21, we will explain a situation where the equivalence between impulse control problem and optimal stopping problem is guaranteed. This is when $\delta = 0$ as illustrated in the sequel.

Corollary 4.21. *For any function* $k \in \mathcal{C}_b(\mathsf{E})$ *satisfying* $\inf_{y \in \mathsf{E}} k(y) = 0$ *and* $e^{-at}k(X(t))$ *is a supermartingale, the value function* $V(x)$ *is the viscosity solution to*

$$\min(aw - \mathcal{A}w, w - k + c) = 0.$$

Proof. Since $e^{-at}k(X(t))$ is a supermartingale, by Lemma 3.34, the function k is a viscosity supersolution to $aw - \mathcal{A}w = 0$. Furthermore, since the function c is a positive function, we have k is also a viscosity supersolution to

$$\min(aw - \mathcal{A}w, w - k(x) + c(x)) = 0,$$

By the comparison principle, the value function $V \leq k$, that is, $\sup_{y \in S}(V(y) - k(y)) \leq 0$. On the other hand since $V \geq 0$ and $\inf_{y \in S} k(y) = 0$, $\sup_{y \in S}(V(y) - k(y)) \geq 0$. Then, $\delta = \sup_{y \in S} V(y) - k(y) = 0$. By (2) in Proposition 4.19, the value function $V = w_\delta$, where w_δ is the viscosity solution to

$$\min(aw - \mathcal{A}w, w - k + c) = 0.$$

■

The above corollary enables us to find an explicit solution of impulse control problem for exponential Lévy process. We show that such problem is equivalent to that of a optimal stopping problem. The following example illustrates an impulse control problem for exponential Lévy process.

Example 4.22. *(Impulse Control of Exponential Lévy process) Assume that the value of a company follows an Exponential Lévy process* Y *without dividend and investment, that is,*

$$Y(t) := e^{X(t)},$$

where $X(t)$ *is Lévy process. We consider the impulse control problem with both dividend payment and investment having fixed costs* $c > 0$*. More precisely, let* $k(x) = x$ *such that* $K(x, y) = x - y - c$*. Then, when* $x > y$*, it means the value of the company decrease from the amount* x *to* y*, that is, the shareholders get the dividend payment* $(x - y)$ *minus fixed cost* c*. Similarly,* $x < y$*, it means the value of company increase from* x *to* y *by capital injection and shareholders pay* $-(x-y-c)$ *totally. Our aim is to maximize the discounted value of the dividends gotten and capital injection paid from shareholders. Its value function function is* *of the form:*

$$V(y) := \sup_{\pi} \boldsymbol{E}^y \Big[\sum_{i=1}^{\infty} e^{-a\tau_i} (Y(\tau_i^-) - Y(\tau_i) - c) \Big]. \tag{4.3.16}$$

Alternatively, we construct an equivalent problem based on the process X such that

$$V(e^x) := \sup_{\pi} \boldsymbol{E}^x \Big[\sum_{i=1}^{\infty} e^{-a\tau_i} (e^{X(\tau_i^-)} - e^{X(\tau_i)} - c) \Big]. \tag{4.3.17}$$

In this case, we replace $k(x) = x$ by $k(x) = e^x$ for X. Using [Mordecki, 2002, Theorem 1] and Corollary 4.21, we can have the following result. Let τ_a be an exponential random variable with parameter $a > 0$. Define the random variable

$$M := \sup_{0 \le t < \tau_a} X(t). \tag{4.3.18}$$

Corollary 4.23. *Suppose that $\boldsymbol{E}\big[e^{X(1)}\big] < e^a$. Then, the value function V defined by* (4.3.16) *is given by*

$$V(x) = \begin{cases} \boldsymbol{E}\Big[\Big(\frac{xe^M}{\boldsymbol{E}\left[e^M\right]} - c\Big)^+\Big] & \text{for } x \in (0, x^*], \\ V(x^*) - x^* + x & \text{for } x \in (x^*, \infty). \end{cases} \tag{4.3.19}$$

where

$$x^* = c\boldsymbol{E}\big[e^M\big]. \tag{4.3.20}$$

Proof. *Define $k_L(x) := exp(x) \wedge L$ for $x \in \mathbb{R}$, where $L > 0$. Since $\boldsymbol{E}\big[e^{X(1)}\big] < e^a$, we have the process $e^{-at+X(t)}$ is a supermartingale and hence $e^{-at+X(t)} \wedge L$ is also a supermartingale. We first compute the value function V_L defined by*

$$\begin{aligned} V(x) &:= \sup_{\pi} \boldsymbol{E}^x \Big[\sum_{i=1}^{\infty} e^{-a\tau_i} (k_L(X(\tau_i^-)) - k_L(X(\tau_i)) - c) \Big] \\ &= \sup_{\pi} \boldsymbol{E}^x \Big[\sum_{i=1}^{\infty} e^{-a\tau_i} (e^{X(\tau_i^-)} \wedge L - e^{X(\tau_i)} \wedge L - c) \Big]. \end{aligned}$$

Since $e^{-at} k_L(X(t))$ is a supermartingale, using Corollary 4.21, we have $\delta = 0$ and *then the value function*

$$V_L(x) = \sup_\tau \boldsymbol{E}^x\big[e^{-a\tau}(e^{X(\tau)} \wedge L) - c\big].$$

As L convergence to the infinity, the value function $V_L(x)$ converges to $V(ln(x))$ pointwisely and we have $V(e^x) = \sup_\tau \boldsymbol{E}^x\big[e^{-a\tau+X(\tau)} - c\big]$. Using [Mordecki, 2002, Theorem 1], the claim is proved. ∎

Remark 4.24. *In this example, we reduce the impulse control problem* (4.3.16) *to American option problem and compute the explicit solution. Consider the optimal impulse control strategy, the capital injection is always not preferable while the dividend should be paid at level x^* with the amount x^*. Intuitively speaking, since the discount rate a is large, the shareholder seek a chance to make the payment of all the money in the company.*

4.3.3 Explicit Solution for one dimension regular diffusion

The above section gives an equivalence between an impulse control problem and an optimal stopping problem. However, in general, $K(x, y)$ defined by (4.3.9) is not commonly used in the actuarial science. Consider a one-dimensional regular diffusion X with state space $\mathsf{E} = [L, R] \subseteq \mathbb{R}$. Define

$$K(x, y) = \begin{cases} k_1(x) - k_1(y) - c_1 \text{ for } x > y \\ k_2(x) - k_2(y) - c_2 \text{ for } x \leq y \end{cases}. \tag{4.3.21}$$

where k_1, k_2 are functions and c_1, c_2 are constants. We are interested in the following impulse control problem instead:

$$V^{(k_{1,2})}(x) := \sup_\pi \boldsymbol{E}^x\left[\sum_{i=1}^\infty e^{-a\tau_i} K(X(\tau_i^-), X(\tau_i))\right] \tag{4.3.22}$$

$$= \sup_\pi \boldsymbol{E}^x\left[\sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-)>X^\pi(\tau_i)}((k_1(X^\pi(\tau_i^-)) - k_1(X^\pi(\tau_i)) - c_1) \right.$$

$$\left. + \sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-)\leq X^\pi(\tau_i)}(k_2(X^\pi(\tau_i^-)) - k_2(X^\pi(\tau_i))) - c_2)\right]. \tag{4.3.23}$$

The above problem often appears in actuarial science and is refereed to as dividend and investment with different proportional costs and fixed costs.

Remark 4.25. *In this situation, the costs of the increasing and decreasing impulse are different. Thus, as compared to the last section, the complexity of the* *problem has emerged. Moreover, for an extreme case that only one side impulse is allowed, we can trivially use our results by setting k_i as a continuous bounded function while $c_i \to \infty$, where $i = 1$ or 2, that is,*

$$K(x,y) = \begin{cases} k_1(x) - k_1(y) - c_1 & \text{for } x > y \\ -\infty & \text{for } x \leq y. \end{cases}$$

For example, this case is applied in the dividend control problem without capital injections.

In order to solve for (4.3.22), we follow the idea described in Beibel and Lerche [2001] with the approach introduced in Chapter 3 in this thesis. Here we assume that X is a regular Feller diffusion, i.e., $\boldsymbol{P}^x\big[\tau_y < \infty\big] > 0$ for all $x \in \mathsf{E}$, where $\tau_y := \inf\{t > 0; X(t) = y\}$. Let $x_0 \in (L, R)$. Define the function

$$\psi_1(x) := \begin{cases} \boldsymbol{E}^x\big[e^{-a\tau_z}\big] & \text{for } x \leq z \\ 1/\boldsymbol{E}^z\big[e^{-a\tau_x}\big] & \text{for } x \geq z \end{cases} \text{ and } \psi_2(x) := \begin{cases} 1/\boldsymbol{E}^z\big[e^{-a\tau_x}\big] & \text{for } x \leq z \\ \boldsymbol{E}^x\big[e^{-a\tau_z}\big] & \text{for } x \geq z \end{cases}. \tag{4.3.24}$$

Here, we separate the points in state space E into two regions:

$$\mathfrak{C} := \{x \in [L,R]; V(x) > \mathcal{M}V(x)\}, \tag{4.3.25}$$
$$\mathfrak{Y} := \{x \in [L,R]; V(x) = \mathcal{M}V(x)\}. \tag{4.3.26}$$

To simplify the problem, we only consider the continuation region $\mathfrak{C}$ which is connected and we distinguish three different cases:

(I) $\mathfrak{C} = (l, r)$,

(II) $\mathfrak{C} = (L, r)$ or $[L, r)$,

(III) $\mathfrak{C} = (l, R)$ or $(l, R]$,

where $L < l < r < R$. Since case (II) and (III) are similar , we only focus on the first two situations which differ on the left hand boundaries. We will show the characterization of the value function $V^{(k_{1,2})}$ defined by (4.3.22).

(I) Our first scenario refers to when $\mathfrak{C} = (l, r)$. It means that when the process reaches l or r, we exercise the impulse strategy which alters the state of the process from l or r to some point inside (l, r).

Proposition 4.26. *Assume that $k_2 - k_1$ is an increasing function, and there exist 4 constants (l, r, p_1, p_2) such that the functions*

$$\begin{aligned} u(x) &:= p_1\psi_1(x) + p_2\psi_2(x) \\ u_i(x) &:= u(x) - k_i(x) \text{ for } i = 1, 2, \end{aligned} \tag{4.3.27}$$

satisfy

1. *u_1 has a local minimum at l and u_2 has a local minimum at r,*
2. *$u_1(r) = \sup_{y\in[l,r]} u_1(y) - c_1$ and $u_2(l) = \sup_{y\in[l,r]} u_2(y) - c_2$.*

Let

$$w_{p_1,p_2,l,r}(x) := \begin{cases} k_2(x) + u(l) - k_2(l) & \text{for } x \in [L, l), \\ u(x) & \text{for } x \in [l, r], \\ k_1(x) + u(r) - k_1(r) & \text{for } x \in (r, R]. \end{cases} \tag{4.3.28}$$

Then the value function satisfies $V^{(k_{1,2})} \geq w_{p_1,p_2,l,r}$, where $V^{(k_{1,2})}$ is defined by (4.3.22).

Furthermore, suppose that

3. *$u_1(y) - u_1(x) \leq c_1$ and $u_2(x) - u_2(y) \leq c_2$ for any $l \leq x < y \leq r$.*
4. *k_2 is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) - a(k_2(l) - u(l)) = 0$ for $x \in [L, l)$ and k_1 is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) - a(k_1(r) - u(r)) = 0$ for $x \in (r, R]$.*

Then, the equality holds, i.e., $V^{(k_{1,2})}(x) = w_{p_1,p_2,l,r}(x)$ for $x \in [L, R]$.

Proof. First, according to Proposition 4.16, the value function $V^{k_{1,2}}$ is the unique viscosity solution to

$$\min(aw(x) - \mathcal{A}w(x), w - \mathcal{M}w) = 0 \text{ for } x \in [L, R]. \tag{4.3.29}$$

Detailed proof is omitted here because it is similar to that of Proposition 4.19. In what follows, we only need to show that the function $w_{p_1,p_2,l,r}$ is also a viscosity solution to (4.3.29).

(1) Let $x \in (l, r)$. Since $w_{p_1,p_2,l,r}(x) = u(x)$ for $x \in (l, r)$, $w_{p_1,p_2,l,r}$ is a viscosity solution to

$$aw(x) - \mathcal{A}w(x) = 0 \text{ for } x \in (l, r).$$

Based on conditions (2) and (3) as well as the construction of $w_{p_1,p_2,l,r}$ by (4.3.28), we also obtain that $w_{p_1,p_2,l,r}(x) \geq \mathcal{M}w_{p_1,p_2,l,r}(x)$ and then (4.3.29) is satisfied.

(2) Let $x \in (l,r)^c$. For $x = l, r$, Condition (1) implies that $w_{p_1,p_2,l,r}$ is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) = 0$. On the other hand, Condition (4) implies $w_{p_1,p_2,l,r}$ is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) = 0$ for $x = [l,r]^c$. Then, to verify (4.3.29), we only need to show $w_{p_1,p_2,l,r}(x) = \mathcal{M}w_{p_1,p_2,l,r}(x)$ which can be gotten from the fact that $k_2 - k_1$ is an increasing function and condition (2). ■

(II) Now we consider two cases when $\mathfrak{C} = (L, r)$ and $\mathfrak{C} = [L, r)$, respectively. For $\mathfrak{C} = (L, r)$, when the process reaches the boundary L or r, the impulse strategy is exercised in the same way as described above; Nevertheless, when $\mathfrak{C} = [L, r)$, the impulse strategy is applied when the process reaches r only. We will demonstrate a similar conclusion which will be reached by an analogous proof as before.

Proposition 4.27. *Let $L < r < R$. Assume that $k_1 - k_2$ is an increasing function, and there exist 3 constants (r, p_1, p_2) such that the functions*

$$\begin{aligned} u(x) &:= p_1\psi_1(x) + p_2\psi_2(x) \\ u_i(x) &:= u(x) - k_i(x) \textit{ for } i = 1, 2, \end{aligned} \tag{4.3.30}$$

satisfy

1. *$p_2 \geq 0$. Furthermore, if $p_2 > 0$, we additionally suppose that $u_2(L) = \sup_{y\in[L,r]} u_2(y) - c_2$ and $\psi_2(L) < \infty$.*
2. *u_1 has a local minimum at r.*
3. *$u_1(r) = \sup_{y\in[L,r]} u_1(y) - c_1$.*

Let

$$w_{p_1,p_2,r}(x) := \begin{cases} u(x) & \textit{for } x \in [L, r], \\ k_1(x) + u(r) - k_1(r) & \textit{for } x \in (r, R]. \end{cases} \tag{4.3.31}$$

Then the value function satisfies $V^{k_{1,2}} \geq w_{p_1,p_2,r}$.

In addition, suppose that

4. *$u_1(y) - u_1(x) \leq c_1$ and $u_2(x) - u_2(y) \leq c_2$ for any $L \leq x < y \leq r$.*
5. *k_1 is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) - a(k_1(r) - u(r)) = 0$ for $x \in (r, R]$.*

Then, the equality holds, i.e., $V^{k_{1,2}}(x) = w_{p_1,p_2,r}(x)$

Proof. It can be proved similarly to the proof of Proposition 4.26. ■

To put it in a more intuitive way, let us illustrate using an example of an Absorbing Feller diffusion on $[0, \infty)$.

Example 4.28. *(Absorbing Feller diffusion on $[0, \infty)$) An absorbing Feller process is a diffusion process with absorbing boundary whose generator is given by*

$$\begin{aligned} D(\mathcal{A}) &:= \{u \in \mathcal{C}_0([a, \infty)) \cap \mathcal{C}^2([a, \infty)); \frac{1}{2}\sigma^2 D_{xx}u(0) + \mu D_x u(0) = 0\}, \\ \mathcal{A}u(x) &:= \frac{1}{2}\sigma^2 D_{xx}u(x) + \mu D_x u(x). \end{aligned} \tag{4.3.32}$$

In this case, ψ_1 and ψ_2 as in (4.3.24) are reduced to

$$\psi_1 = e^{l_1}x - e^{l_2}x \text{ and } \psi_2 = e^{l_2}x \tag{4.3.33}$$

where $l_1 = \frac{-\mu+\sqrt{\mu^2+2a\sigma^2}}{\sigma^2}$ and $l_2 = \frac{-\mu-\sqrt{\mu^2+2a\sigma^2}}{\sigma^2}$.

Recall that we are interested in the following impulse control problem.

$$\begin{aligned} V(x) := \sup_\pi \boldsymbol{E}^x \Bigg[& \sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) > X^\pi(\tau_i)}((k_1(X^\pi(\tau_i^-)) - k_1(X^\pi(\tau_i)) - c_1) \\ & + \sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) \le X^\pi(\tau_i)}(k_2(X^\pi(\tau_i^-)) - k_2(X^\pi(\tau_i))) - c_2) \Bigg]. \end{aligned} \tag{4.3.34}$$

Such problem was solved in Guan and Liang [2014] for $k_1(x) = \beta_1 x$ and $k_2(x) = \beta_2 x$. Besides, in their work, they analysed a combined impulse and stochastic control problem. Here, we also study this problem with a focus on the impulse control side when k_1 and k_2 are of a general form which includes the exponential type (see below). We present three different scenarios as described below.

1. *Liner form: $k_1^{(l)}(x) = \beta_1 x$ and $k_2^{(l)}(x) = \beta_2 x$.*
2. *Exponential form: $k_1^{(e)}(x) = \beta_1 e^x$ and $k_2^{(e)}(x) = \beta_2 e^x$.*
3. *Quadratic form: $k_1^{(q)}(x) = \beta_1 x^2 + \gamma_1 x$ and $k_2^{(q)}(x) = \beta_2 x^2 + \gamma_2 x$.*

where $\beta_1 \le \beta_2$ and $\gamma_1 \le \gamma_2$. In finance, the first one can be used to study dividends and investment problems. The second form can be applied to processes written in exponential forms, which we have in fact discussed before in Example 4.22. The last situation can be found in Ohnishi and Tsujimura [2006] to deal with the impulse control problem with quadratic costs. Here, although the functions k_1 and

k_2 are not bounded in any of the aforementioned cases, we analogously suppose the value function of these cases can be obtained based on the convergence of the value function with $k_1 \wedge L$ and $k_2 \wedge L$, as $L \to \infty$.

Note that we have mentioned the exponential case in Example 4.22. As a consequence from Corollary 4.23, the form of $\mathfrak{C}$ is of the type $\mathfrak{C} = [L, r)$. Results should be very similar to the ones under the exponential case, so we do not duplicate our discussions. In the sequel, we illustrate a linear case for an absorbing BM.

Corollary 4.29. *Let X be an absorbing BM whose generator is written as* (4.3.32) *and the value function V of X is defined by* (4.3.22)*, where $k_1 = \beta_1 x$ and $k_2 = \beta_2 x$ with $\beta_2 > \beta_1 > 0$. Let ψ_1 and ψ_2 represented in* (4.3.33).

1. *Given $\mu \leq 0$, assume there exists $c \in \mathbb{R}$ and $x^* \in (0, \infty)$ such that*

   (a) *$c\psi_1 - \beta_1 x$ has a local minimum at x^* .*

   (b) *$c_1 = c\psi_1(x^*) - \beta_1 x^*$.*

   (c) *$c\psi_1(x) - \beta_1 x$ is decreasing in $[L, x^*]$.*

   Then, the value function is

$$V(x) = \begin{cases} c\psi_1(x) & \text{for } x \in [0, x_*], \\ k_1(x) - k_1(x^*) + c\psi_1(x) & \text{for } x \in (x_*, \infty) \end{cases} \tag{4.3.35}$$

2. *Given $\mu > 0$, assume there exist (p_1, p_2, x^*) such that*

   (a) *$p_1 \in \mathbb{R}$, $p_2 > 0$.*

   (b) *$p_1\psi_1(x) + p_2\psi_2(x) - k_1(x)$ has a local minimum at x^*.*

   (c) *$p_1\psi_1(x^*) + p_2\psi_2(x^*) - \beta_1 x^* = \max_{x\in[0,x^*]}(p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x) - c_1 = p_1\psi_1(x_r) + p_2\psi_2(x_r) - \beta_1 x_r$ where $x_r \in [0, x^*]$. $p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$ is increasing in $[0, x_r]$ and decreasing in $[x_r, x^*]$.*

   (d) *$p_1\psi_1(0) + p_2\psi_2(0) = \max_{x\in[0,x^*]}(p_1\psi_1(x) + p_2\psi_2(x) - \beta_2 x) - c_2 = p_1\psi_1(x_r) + p_2\psi_2(x_l) - k_2 x_l$, where $x_l \in [0, x^*]$. Additionally, $p_1\psi_1(x) + p_2\psi_2(x) - \beta_2 x$ is increasing in $[0, x_l]$ and decreasing in $[x_l, x^*]$.*

   Then, the value function is

$$V(x) = \begin{cases} p_1\psi_1(x) + p_2\psi_2(x) & \text{for } x \in [0, x^*], \\ k_1(x) - k_1(x^*) + p_1\psi_1(x^*) + p_2\psi_2(x^*) & \text{for } x \in (x^*, \infty). \end{cases} \tag{4.3.36}$$

Proof.

Theses results can be proved by Proposition 4.27, where claim (1) is from the case $[L, r)$ and claim (2) is from the case (l, r). ■

Next we present a numerical result to explain in details to get an idea of what we could obtain from the above theorem. Parameter values are $\beta_1 = 0.9$, $\beta_2 = 1.5$, $c_1 = 2$ and $c_2 = 4$.

(1)The first case tells us when $\mu = -1$ and $\sigma = 1$, $\mathfrak{C}$ has a form $[L, r)$. In addition, one can compute $\psi_1(x) = e^{2.0488x} - e^{-0.0488x}$ and $\psi_2(x) = e^{-0.0488x}$, as well as deriving $c = 0.0017$ and $x^* = 2.71$. Based on these values, we plotted the function u_1 in the figure.

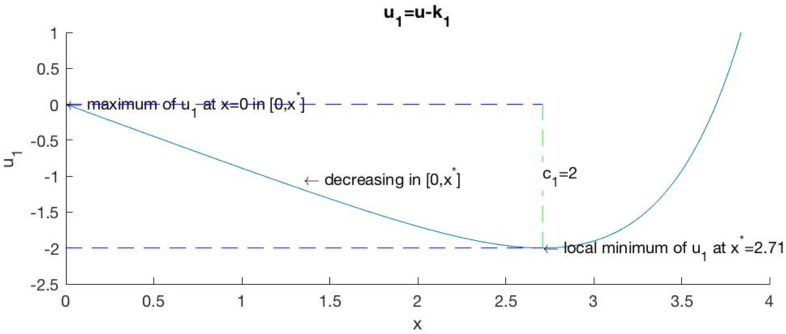


Figure 4.1: This graph sketches $u_1(x) = c\psi_1(x) - \beta_1 x$. Since ψ_1 is convex, it has a minimum at $x^* = 2.71$. Additionally, we can see it decreasing from 0 to $x^* = 2.71$ and then increasing. Hence, the maximum of u_1 in $[0, x^*]$ is at $x = 0$. That is to say, the condition 1(b) is satisfied under $u_1(0) - u_1(x^*) = c_1$.

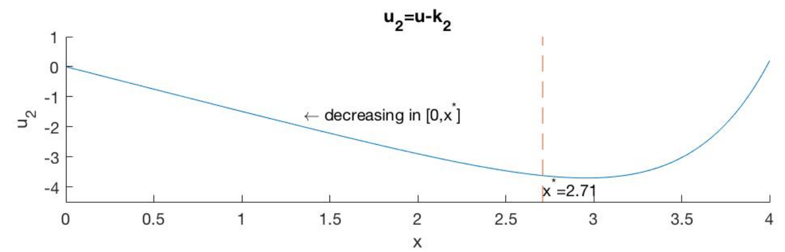


Figure 4.2: This graph sketches $u_2(x) = c\psi_1(x) - \beta_2 x$. Since u_1 is decreasing from 0 to x^* and $\beta_2 \geq \beta_1$, then u_2 is decreasing in $[0, x^*]$. Thus, since $c_2 > 0$, it is true that $u_2(x) - u_2(y) \leq c_2$ for any $x > y$.

Intuitively, we could as a result propose an optimal strategy. It suggests an impulse from $x^* = 2.71$ to 0 when the process reaches $x^* = 2.71$.

(2) As opposed to (1), the second scenario considers all the same parameter values except for $\mu = 1$. This refers to the situation where $\mathfrak{C}$ has a form (L, r). Consequently, $\psi_1(x) = e^{0.0488x} - e^{-2.0488x}$ and $\psi_2(x) = e^{-2.0488x}$. One can further find out $p_1 = 10.01$, $p_2 = 4.33$ and $r = 12$.

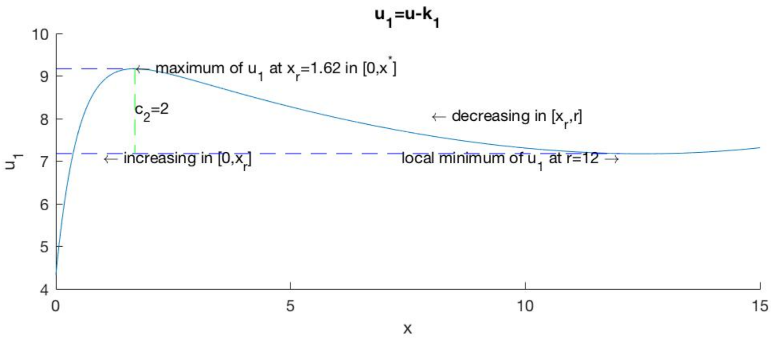


Figure 4.3: This graph sketches $u_1(x) = p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$. As it can be seen, u_1 has a local maximum at $x_r = 1.62$ and a local minimum at $r = 12$. Additionally, u_1 is increasing in $[0, x_r]$ and is decreasing in $[x_r, x_*]$. The condition we need to impose here is that $u_1(x_r) - u_1(r) = c_2$.

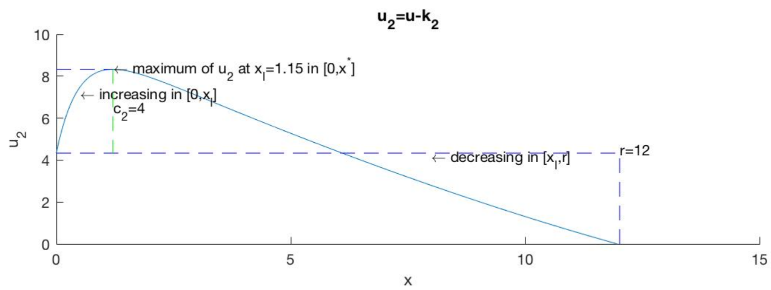


Figure 4.4: This graph sketches $u_2(x) = p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$. Again, u_1 has a local maximum at $x_l = 1.15$. It entitles that u_1 is increasing in $[0, x_r]$ and is decreasing in $[x_r, x_*]$. We need to impose the condition $u_1(x_r) - u_1(0) = c_2$.

Again intuitively, the desired strategy is to carry out an impulse from $r = 12$ to $x_r = 1.62$ when the process reaches $r = 12$. Meanwhile, it advises to exercise an impulse from 0 to $x_l = 1.16$ when the process reaches 0.

Comparing these two strategies, we found that when μ is positive and large enough, the increasing impulse is more effective to exercise, since the positive drift μ indicates an uptown of the asset for a company, and vice versa. In the opposite situation, we should just use one decreasing impulse strategy to the absorbing point to get the value efficiently.

4.4 Perturbation and Application

In the above section, we mainly discussed the specific case for the impulse control optimal whose operator is $\mathsf{G}u = \sup_{y\in\mathsf{E}}(u(y) + K(x, y))$. In this section, we aim to analyse a series of problems considering the operator F given by perturbation instead.

More precisely, the construction of the Feller semigroup based on the perturbation relies on Hille-Yosida theorem (see Lemma 3.49).

It allows us to verify the existence of Feller process with large jumps (see for example [Böttcher et al., 2013, Section 4.3.] and [Taira, 2004, Corollary 9.51.]). To this end, let b be a non-negative function in $\mathcal{C}_b(\mathsf{E})$, λ be a non-negtive constant and $\mathcal{B}$ be a linear operator on $\mathcal{C}_b(\mathsf{E})$. Then we can define the perturbation operator $\mathcal{A}_{pb} : B(\mathsf{E}) \to B(\mathsf{E})$ by

$$\mathcal{A}_{pb}w(x) := b(x)\mathcal{B}w(x) - \lambda b(x)w(x) \text{ for } x \in \mathsf{E}, w \in B(\mathsf{E}). \tag{4.4.1}$$

To construct the process with perturbation, we make the following assumptions on the operator $\mathcal{B}$.

Assumption 5.

1. *$\mathcal{B}$ is a linear operator and $\mathcal{B} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.*

2. *$\mathcal{B}$ is positive and bounded with $\lambda \geq \|\mathcal{B}\|_\infty$.*

Following Lemma 3.49, we present this lemma.

Lemma 4.30. *Suppose that Assumption 5 holds and $\mathcal{B} : \mathcal{C}_0(\mathsf{E}) \to C_0(\mathsf{E})$. Let $(\mathcal{A}_0, D(\mathcal{A}_0))$ be the generator of some Feller process. Then, $(\mathcal{A}_0 + \mathcal{A}_{pb}, D(\mathcal{A}_0))$ is also the generator of some Feller semigroup.*

Proof. One can check the positive maximum property of $\mathcal{A}_{pb}$ and $\mathcal{A}_{pb} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$. ■ Now let $(\mathcal{A}_0, D(\mathcal{A}_0))$ be a Feller semigroup and X be a Feller process with the infinitesimal generators $(\mathcal{A}_0 + \mathcal{A}_{pb}, D(\mathcal{A}_0))$. We are interested in the optimal stopping problem

$$V(x) := \sup_\tau \boldsymbol{E}^x\left[\int_0^t e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau} g(X(\tau))\right]. \tag{4.4.2}$$

By Theorem 4.10, we have the following proposition to characterise the value function V in the viscosity sense.

Proposition 4.31. *Suppose that Assumption 5 holds and $(\mathcal{A}_0, D(\mathcal{A}_0))$ is a generator of a Feller process. If X is a Feller process with infinitesimal generator*

$(\mathcal{A}+\mathcal{A}_{pb}, D(\mathcal{A}))$*, then the value function defined by* (4.4.2) *is the unique viscosity solution* $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$\min(aw - \mathcal{A}_0 w - (\mathcal{A}_{pb} w + f), w - g) = 0. \tag{4.4.3}$$

Proof. We first show that there exists a unique viscosity solution to (4.4.3). After transforming (4.4.3) using $\mathcal{A}_{pb} u := b\mathcal{B}u - \lambda bu$, its viscosity solution is equivalent to that of

$$\min((a + \lambda b)w - \mathcal{A}_0 w - (b\mathcal{B}w + f), w - g) = 0. \tag{4.4.4}$$

Since $a + \lambda b \in \mathcal{C}_b(\mathsf{E})$ and $a + \lambda b > 0$, this is further equivalent to the viscosity solution to

$$\min(w - \frac{1}{a + \lambda b}\mathcal{A}_0 w - \frac{b\mathcal{B}w + f}{a + \lambda b}, w - g) = 0. \tag{4.4.5}$$

To show the properties of the viscosity solution to (4.4.5), we need the definitions of F and G in order to apply Theorem 4.10.

$$\mathsf{F}u := \frac{b\mathcal{B}u + f}{a + \lambda b} \text{ and } \mathsf{G}u := g. \tag{4.4.6}$$

Under such definition, let us first verify that Assumption 2 and Assumption 3 hold.

(i) Since $\mathcal{B}$ is defined from $\mathcal{C}_b(\mathsf{E})$ to itself and $b \in \mathcal{C}_b(\mathsf{E})$, we have F is defined from $C_b(\mathsf{E})$ to itself.

(ii) The monotonic property of F in Assumption 2 follows from the fact that $\mathcal{B}$ is positive, $a + \lambda b > 0$ and $b \geq 0$.

(iii) The convexity of F in Assumption 2 follows from the linearity of $\mathcal{B}$, that is,

$$\begin{aligned}\mathsf{F}(pu_1 + (1-p)u_2) &= \frac{b\mathcal{B}(pu_1 + (1-p)u_2) + pf + (1-p)f}{a + \lambda b} \\ &= p\mathsf{F}u_1(x) + (1-p)\mathsf{F}u_2(x).\end{aligned}$$

(iv) Let $\kappa > 0$ be a constant and $w_+ := \max(\frac{\|f\|_\infty}{a} + (a + \lambda\|b\|_\infty)\kappa, \|g\|_\infty + \kappa)$

be a constant function. Then,

$$
\begin{aligned}
&\min(w_+ - \frac{1}{a+\lambda b}\mathcal{A}_0 w_+ - \frac{b\mathcal{B}w_+ + f}{a+\lambda b} - \kappa, w_+ - g - \kappa) \\
=&\min(\frac{aw_+}{a+\lambda} - \frac{\mathcal{A}_0 w_+ + b\mathcal{B}w_+ + \lambda w_+}{a+\lambda b} - \frac{f}{a+\lambda b} - \kappa, w_+ - g - \kappa) \\
=&\min(\frac{aw_+ - f}{a+\lambda b} - \kappa, w_+ - g - \kappa) \geq 0.
\end{aligned}
$$

Hence, using Lemma 4.12, we have (1) in Assumption 3.

(v) (2) in Assumption 3 is true, since $\mathsf{F}(w+C) - \mathsf{F}w = \frac{b\mathcal{B}C}{a+\lambda} \leq \frac{C\lambda b}{a+\lambda b} \leq 1$.

This verification for G is more straightforward and is omitted here. Thus, we can conclude that there exists a unique viscosity solution to (4.4.3) by Theorem 4.10.

Next, we prove that the value function V defined by (4.4.2) is the unique viscosity solution to (4.4.3). Since X is a Feller process, the value function V defined by (4.4.2) is the unique viscosity solution to

$$\min(aw - \mathcal{A}w - f, w - g) = 0. \tag{4.4.7}$$

where $\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_{pb}$. Then, since the viscosity solution to 4.4.7 is unique, we only need to prove that the viscosity solution w to (4.4.3) is also the viscosity solution to (4.4.7). Let w be the viscosity solution to (4.4.5) which is equivalent with the viscosity solution to (4.4.3). Assume that $\phi \in D(\mathcal{A}_0)$ satisfies $\phi - w$ has a global minimum at $x_0 \in \mathsf{E}$ such that $\phi(x_0) = w(x_0)$. Since w is a viscosity subsolution to (4.4.5), we have

$$\min(\phi(x_0) - \frac{1}{a+\lambda b}\mathcal{A}_0\phi(x_0) - \mathsf{F}w(x_0), \phi(x_0) - g(x_0)) \leq 0.$$

In addition, since $\phi \geq w$ and F is increasing, we have $\mathsf{F}\phi \geq \mathsf{F}w$ and then

$$\min(\phi(x_0) - \frac{1}{a+\lambda b}\mathcal{A}_0\phi(x_0) - \mathsf{F}\phi(x_0), \phi(x_0) - g(x_0)) \leq 0.$$

This is the same as

$$\min(a\phi(x_0) - \mathcal{A}\phi(x_0) - f(x_0), \phi(x_0) - g(x_0)) \leq 0.$$

Therefore, w is also a viscosity subsolution to (4.4.3). The case of the viscosity supersolution can be proved similarly. ∎

Now that we have proved the existence and uniqueness of the viscosity solution. We subsequently construct a numerical scheme to derive the value function in the next proposition.

Proposition 4.32. *Suppose the same assumptions hold as in Proposition 4.31. Let $v_0 \in \mathcal{C}_b(\mathsf{E})$ be a viscosity subsolution to*

$$\min(aw - \mathcal{A}_0 w - (\mathsf{F}_{pb} w + f), w - g) = 0. \tag{4.4.8}$$

Let v_n be the viscosity solution to

$$\min(aw - \mathcal{A}_0 w - \mathsf{F}_{pb}(v_{n-1} + f), w - g) = 0, \tag{4.4.9}$$

or equivalently,

$$v_n(x) := \sup_\tau \boldsymbol{E}^x \left[\int_0^\tau e^{-s} \frac{b(Y(s))\mathcal{B}v_{n-1}(Y(s)) + f(Y(s))}{a + \lambda b(Y(s))} \mathrm{d}s + e^{-\tau} g(Y(s)) \right], \tag{4.4.10}$$

where Y is a Feller process with the infinitesimal generator $(\frac{\mathcal{A}_0}{a+\lambda b}, D(\mathcal{A}_0))$. Then v_n converges uniformly to the viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ to (4.4.3).

Proof. Since we have proved the value function is the viscosity solution to (4.4.7). Then, we can transform our problem to an iterative optimal stopping method by Theorem 4.13. ∎

Next, we present three examples that satisfy Assumption 5 : jump processes, regime switching Feller processes and semi-Markov processes. We recall that the iterative optimal stopping method was also used in Babbin et al. [2014] for regime switching and Bayraktar and Xing [2009] for pricing of the American option for jump processes. Results obtained from our method are consistent with theirs.

4.4.1 Compound Poisson operator

Let E denote a measurable space with a positive bounded rate kernel $\alpha(x, \cdot)$. In this part, we consider the case when

$$\lambda = 1, b(x) = \int_{\mathsf{E}} \alpha(x, \mathrm{d}y) \text{ and } \mathcal{B}u(x) := \frac{\int_{\mathsf{E}} u(y)\alpha(x, \mathrm{d}y)}{b(x)} \text{ for any } u \in B(\mathsf{E}).$$

Then the operator $\mathcal{A}_{pb}$ can be written as

$$\mathcal{A}_{pb}u(x) := b(x)\mathcal{B}u(x) - \lambda b(x) = \int_{\mathsf{E}} (u(y) - u(x))\alpha(x, \mathrm{d}y). \tag{4.4.11}$$

This operator can be seen as the generator of pseudo-Poisson process (see for example [Kallenberg, 2006, Proposition 17.2]). In order to make sure Assumption 5 is true, we need the kernel $\alpha(x, \cdot)$ to satisfy

$$v(x) = \int_{\mathsf{E}} u(x)\alpha(x, \mathrm{d}y) \in \mathcal{C}_b(\mathsf{E}) \text{ for any } u \in C_b(\mathsf{E}). \tag{4.4.12}$$

We first consider a simple case introduced in Bayraktar and Xing [2009]. It focuses on an optimal stopping problem for pricing American options. Its value function is defined by

$$V^{(c)}(x) := \sup_{\tau} \boldsymbol{E}^x \big[e^{-a\tau}(K - e^{X(\tau)})^+ \big],$$

where X is a jump diffusion, i.e.,

$$X(t) = (\mu - \frac{1}{\sigma^2})t + \sigma W(t) + \sum_{n=1}^{N(t)} S_n,$$

and $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with intensity $\lambda_0 > 0$ and $\{S_n\}_{n\in\mathbb{N}}$ represents a sequence of independent and identical random variables. Here, X is a Lévy process with the infinitesimal generator

$$\begin{aligned} &D(\mathcal{A}) := \mathcal{C}^2_*(\mathbb{R}) \\ &\mathcal{A}u(x) := (\mu - \frac{1}{\sigma^2})D_x u(x) + \frac{1}{2}\sigma^2 D_{xx}u(x) + \int_{\mathbb{R}} (u(x+y) - u(x))\lambda F(\mathrm{d}y), \end{aligned}$$

where F is the distribution of S_n. In this way, we can decompose the infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ by

$$\begin{aligned} &\mathcal{A}_0 u(x) := (\mu - \frac{1}{\sigma^2})D_x u(x) + \frac{1}{2}\sigma^2 D_{xx}u(x) \text{ for } u \in D(\mathcal{A}_0) := \mathcal{C}^2_*(\mathbb{R}), \\ &\mathsf{F}_{bp}u(x) := \int_{\mathbb{R}} (u(x+y) - u(x))\alpha(x, \mathrm{d}y), \end{aligned}$$

where $\alpha(x, \mathrm{d}y) := \lambda F(\mathrm{d}y)$ such that $(\mathcal{A}, D(\mathcal{A})) = (\mathcal{A}_0 + \mathsf{F}_{bp}, D(\mathcal{A}_0)$. Then, according to Proposition 4.32, we can use the following corollary to derive the corresponding value function $V^{(c)}$.

Corollary 4.33. *Let $v_0(x) := (K - e^x)^+$. Define*

$$v_n(x) := \boldsymbol{E}^x \Bigg[\int_0^{\tau} e^{-s} \frac{\lambda_0}{a + \lambda_0} \big(\int_{\mathbb{R}} v_{n-1}(Y(s) + y) F(\mathrm{d}y) \big) \mathrm{d}s + e^{-\tau}(K - e^{Y(\tau)})^+ \Bigg],$$

where Y is a diffusion defined by $Y(t) = \frac{(\mu - \frac{1}{\sigma^2})}{a+\lambda_0}t + \frac{\sigma}{a+\lambda_0}W(t)$. Then, the sequence *of functions $\{v_n\}_{n\in\mathbb{N}}$ converges to the value function $V^{(c)}$ uniformly.*

Remark 4.34. *Notice that results computed by the proposed iterative optimal stopping method in Corollary 4.33 coincides with those in [Bayraktar and Xing, 2009, Section 3].*

4.4.2 Regime Switching Process

The second example is an extension from Babbin et al. [2014] where regime switching diffusion processes were studied. We generalize the underlying processes to regime switching Feller processes by adding a perturbation operator.

Here, $\mathcal{S} := \{1, 2, \ldots, N\}$ is a finite discrete space, where N is a positive integer. Let $(\mathcal{A}_i, D(\mathcal{A}_i))$ be the infinitesimal generators of some Feller semigiroups on $\mathcal{C}_0(\mathsf{E})$. Then, define the operator $(\mathcal{A}_0^{(r)}, D(\mathcal{A}_0^{(r)}))$ as follows:

$$\begin{aligned} D(\mathcal{A}_0^{(r)}) &:= \{u \in \mathcal{C}_0(S \times \mathsf{E}); u(i, \cdot) \in D(\mathcal{G}_i)\}, \\ \mathcal{A}_0^{(r)} u(i, x) &:= \mathcal{A}_i u_i(x) \text{ for } i \in S \text{ and } x \in \mathsf{E}, \end{aligned} \tag{4.4.13}$$

where $u_i(x) := u(i, x)$. By Hille-Yosida theorem, the above generator is the infinitesimal generator of some Feller semigroup. Additionally, we introduce a bounded operator

$$\mathcal{A}_{pb}^{(r)} u(i, x) := \sum_{j \in N} q_{ij}(x)(u(j, x) - u(i, x)), \tag{4.4.14}$$

where $q_{ij} \in \mathcal{C}_b(\mathsf{E})$ and $q_{ij} \geq 0$. Since $\mathcal{A}_{pb}^{(r)}$ satisfies the positive maximum principle and $\mathcal{A}_{pb}^{(r)} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$, the operator $((\mathcal{A}_0^{(r)} + \mathcal{A}_{pb}^{(r)}, D(\mathcal{A}^{(r)})))$ is the infinitesimal generator of some Feller semigroup.

Then, there exists a corresponding Feller process $(I(s), X(s))$ with state space $\mathcal{S} \times \mathsf{E}$ whose infinitesimal generator is $(\mathcal{A}_0^{(r)} + \mathcal{A}_{pb}^{(r)}, D(\mathcal{A}^{(r)}))$.

It allows us to combine Feller processes with the same state spaces but different behaviours relying on a Markov chain. One example of such processes can be also found in Section 3.7.3.

Therefore, our interest lies in the optimal stopping problem of the Feller process $(I(s), X(s))$:

$$V^{(r)}(i, x) := \sup_{\tau} \boldsymbol{E}^{i,x}\left[\int_0^{\tau} e^{-as} f(I(s), X(s))\mathrm{d}s + e^{-a\tau} g(I(\tau), X(\tau))\right]. \tag{4.4.15}$$

Again we can derive its value function using the iterative optimal stopping method below.

Corollary 4.35. *Let $v_0(i,x) := g(i,x)$. Define*

$$v_n(i,x) = \sup_\tau \boldsymbol{E}^x \Big[\int_0^\tau e^{-s} \big(\frac{f(i, Y^{(i)}(s))}{a + \sum_{j \in N} q_{ij}(x)} - \sum_{j \in N} \frac{q_{ij}(x) v_{n-1}(j,x)}{a + \sum_{j \in N} q_{ij}(x)} \big) \mathrm{d}s + e^{-\tau} g(i, Y^{(i)}(\tau)) \Big]$$

for $n \geq 1$, where $Y^{(i)}$ is the a process with the generator $(\frac{1}{a+\sum_{j\in N} q_{ij}(x)} \mathcal{A}_i, D(\mathcal{A}_i))$. Then, the value function v_n converges to the value function $V^{(r)}$ defined by (4.4.15) uniformly.

Proof. The proof follows from Proposition 4.32. ■

Remark 4.36. *In fact, a similar result can be seen in Babbin et al. [2014] whose model setting is for regime switching diffusions only.*

4.4.3 Semi-Markov process

Finally, we study an application to optimal stopping problems for semi-Markov processes. As far as we are concerned, this problem has not been solved using viscosity methods in literature.

Let us illustrate this by the following example. Consider a risk process

$$X(t) := x + t - \sum_{n=1}^{N^{(s)}(t)} S_n,$$

where $N^{(s)}(t)$ is a renewal process with inter-arrival time $\{T_n\}_{n\in\mathbb{N}}$ conforming to the distribution law F_T, and $\{S_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d random variables with a distribution function F_S.

Let $\xi(t)$ be the time from the last jump and $Y := \{\xi(t), X(t)\}_{t\geq 0}$ be a Markov process. This example can be also found in the example in Section3.6.2.2 by adding a drift. Its infinitesimal generator $(\mathcal{A}^{(s)}, D(\mathcal{A}^{(s)}))$ is

$$\begin{aligned} D(\mathcal{A}^{(s)}) &:= \{u \in \mathcal{C}_0([0,\infty] \times \mathbb{R}); u \text{ is first differentiable and } \frac{\partial}{\partial \xi} u(\infty, x) = 0\} \\ \mathcal{A}^{(s)} u(\xi, x) &:= \frac{\partial}{\partial \xi} u(\xi, x) + \frac{\partial}{\partial x} u(\xi, x) + s(\xi) \int_{\mathbb{R}} (u(0, x+\zeta) - u(\xi, x)) \mathrm{d}F(\zeta), \end{aligned} \tag{4.4.16}$$

where the function s is the hazard function of the distribution F_T introduced in Section3.6.2.2.

Then, we decompose the generator $\mathcal{A}^{(s)}$ via

$$\mathcal{A}_0^{(s)}u(\xi,x) := \frac{\partial}{\partial\xi}u(\xi,x) + \frac{\partial}{\partial x}u(\xi,x), \tag{4.4.17}$$

$$\mathcal{A}_{pb}^{(s)}u(\xi,x) := s(\xi)\int_{\mathbb{R}}(u(0,x+\zeta)-u(\xi,x))\mathrm{d}F(\zeta). \tag{4.4.18}$$

Here, we will demonstrate numerical approximation results deduced by the iterative optimal stopping method. To this end, the value function $V^{(s)}$ can be written as

$$V^{(s)}(x) := \sup_{\tau}\boldsymbol{E}^{(0,x)}\left[e^{-a\tau}g(X(\tau))\right] \tag{4.4.19}$$

Under this setting, we can have similar conclusions as before.

Proposition 4.37. *Assume that $g \in \mathcal{C}_b(\mathbb{R})$.*

1. *The value function $V^{(s)}(x) = w(0,x)$ for $x \in \mathbb{R}$, where w is the unique viscosity solution $w \in \mathcal{C}_b([0,\infty]\times\mathbb{R})$ to*

$$\min(aw - \mathcal{A}_0^{(s)}w - \mathcal{A}_{pb}^{(s)}w, w-\bar{g}) = 0. \tag{4.4.20}$$

   where $\bar{g}(\xi,z) = g(x)$ for $y \in [0,\infty]$ and $z \in \mathbb{R}$.

2. *Let $v_0(y,x := \bar{g}$. Define v_n as the viscosity solution in $w \in \mathcal{C}_b([0,\infty]\times\mathbb{R})$ to*

$$\min(aw - \mathcal{A}_0^{(s)}w - \mathcal{A}_{pb}^{(s)}v_{n-1}, w-\bar{g}) = 0, \tag{4.4.21}$$

   or equivalently,

$$\begin{aligned} v_n(y,z) := \sup_{\tau}\boldsymbol{E}^{(y,z)}\Bigg[&-\int_0^{\tau} e^{-\int_0^s(a+s(Y(l)))\mathrm{d}l}s(Y(s))\int_{\mathbb{R}}u(0,Z(s)+\zeta)\mathrm{d}F(\zeta)\mathrm{d}s \\ &+ e^{-\int_0^{\tau}(a+s(Y(l)))\mathrm{d}l}\bar{g}(Y(\tau),Z(\tau))\Bigg], \end{aligned}$$

   where $\{Y(t),Z(t)\}_{t\geq 0}$ is a Feller process with generator $(\mathcal{A}_0^{(s)}, D(\mathcal{A}_0^{(s)}))$. Then, $v_n(0,\cdot)$ converges to the value function $V^{(s)}$ uniformly.

Proof. The proof follows from Proposition 4.32. ■

Remark 4.38. *Based on (2) in Proposition 4.37, the optimal stopping problem for semi-Markov process can be analogously solved by constructing an iterative optimal stopping problem for two-dimensional deterministic processes.*

Specifically, set T_n as a mixture exponential distribution and S_n as an exponential distribution,i.e.,

$$\begin{aligned}F_T(x) &:= 1 - \beta e^{-\lambda_1 x} - (1-\beta)e^{-\lambda_2 x},\\ F_S(x) &:= 1 - e^{-\gamma x},\end{aligned}$$

where $\beta \in [0,1]$ is the weight, $\lambda_1, \lambda_2, \gamma$ are three positive parameters. Then, the force rate of the inter-arrival time is

$$s_\beta(y) = \frac{\beta\lambda_1 e^{-\lambda_1 y} + (1-\beta)\lambda_2 e^{-\lambda_2 y}}{\beta e^{-\lambda_1 y} + (1-\beta)e^{-\lambda_2 y}}.$$

The value function under concern here is

$$V^{(\beta)} := \sup_\tau \boldsymbol{E}^{0,x}\big[e^{-a\tau}(X(\tau)) \vee 0) \wedge L\big],$$

which can characterized by a viscosity solution

$$\begin{aligned}\min(&aw(\xi,x) - \mathcal{A}_0^{(s)} w(\xi,x) - s_\beta(\xi)\int_{\mathbb{R}^+}(u(0,x-\zeta) - u(\xi,x))\lambda e^{-\lambda\zeta}\mathrm{d}\zeta,\\ &w(\xi,x) - (x\vee 0)\wedge L) = 0.\end{aligned}$$

Here we derive a numerical solution for such problem. For this question, $\bar{g}$ in (4.4.20) is settled by $\bar{g}(y,z) := (y \vee 0) \wedge c$. We are able to solve the value function numerically using the iterative optimal stopping method. As a consequence, we sketch both the value function and exercise boundaries under different scenarios based on various choices of β.

Now we present some numerical results with pre-determined parameters $\lambda_1 = 1$, $\lambda_2 = 3$, $\gamma = 1$, discount rate $a = 0.25$ and $L = 2$ The rate β can take values from 0 to 1.

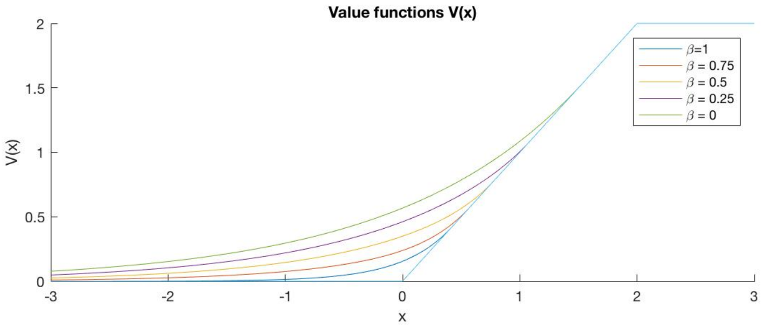


Figure 4.5: Since the hazard rate of F_T increases with the increase of β, then the frequency of the negative jumps increases. Besides, since the payoff g function is an increasing function, intuitively speaking, the value function $V^{(\beta)}$ increases with β as can be see from the figure.

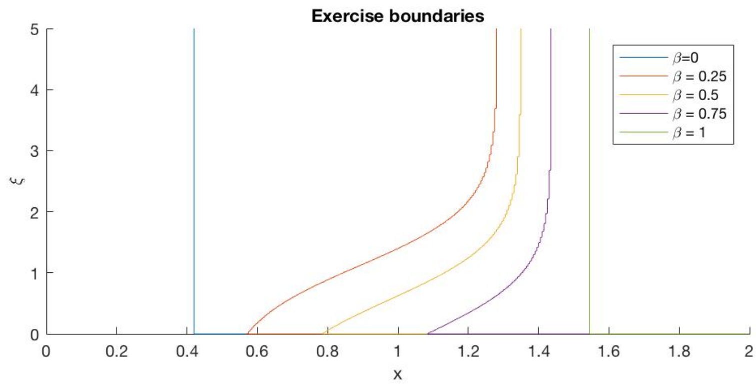


Figure 4.6: Each line represents the boundary of an exercise. We should stop when $\{\xi(t), X(t)\}_{t\geq 0}$ hit the left hand side of the line. We can see that for each $\beta \in (0, 1)$, when the time from the last jump ξ continues to grow, we will stop at rising levels of the state x based on process X. However, when $\beta = 0$ or $\beta = 1$, since the process X is Markov, the optimal stopping strategy does not depend on the time ξ.

4.5 Non-negative Random discount

In previous sections, the discount rate a is a positive constant. This guarantees the value function to be finite for infinite horizon stopping problems. On the contrary, the aim of this section is to relax the assumption on the discount rate. In this section, the discount rate is treated as a random variable. We start by studying the properties of the value function

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} e^{-\int_0^s r(X(s))\mathrm{d}s} f(X(s))\mathrm{d}s + e^{-\int_0^{\tau} r(X(s))\mathrm{d}s} g(X(\tau)) \right], \quad (4.5.1)$$

where $r \in \mathcal{C}_b(\mathsf{E})$ is a random nonegative discount rate and $f, g \in \mathcal{C}_b(\mathsf{E})$. It is worth mentioning that the discount rate r we assumed here is not necessarily uniformly separated from 0. For example, the work Palczewski and Stettner [2014] considers an optimal stopping problem for non-uniformly ergodic Feller-Markov processes. The authors proved the continuity of the value function and its variational characterisation in the viscosity sense, that is, they showed that it is a viscosity solution to

$$\min(rw - \mathcal{A}w - f, w - g) = 0. \quad (4.5.2)$$

Note, however, that they did not prove the uniqueness of the viscosity solution. Here, we provide the proof of the uniqueness of the viscosity solution as a consequence of Theorem 4.10. Notice that the ergodic property (see Palczewski and Stettner [2014]) of the Feller process is not necessary in our proof for the uniqueness. Instead, we make the following assumption.

Assumption 6. *There exist $\kappa > 0$ and $w_+ \in \mathcal{C}_b(\mathsf{E})$ such that w_+ is a viscosity supersolution to*

$$rw - \mathcal{A}w - f - \kappa = 0. \quad (4.5.3)$$

This is a reasonable assumption for common problems encountered in literature. For instance, suppose that r is a continuous bounded function that $\inf_{x \in \mathsf{E}} r(x) = a > 0$. For this case, we can choose $w_+ = \frac{\|f\|_\infty + 1}{a}$ and $\kappa = 1$ so that

$$rw_+ - \mathcal{A}w_+ - f - \kappa \geq \|f\|_\infty + 1 - f - 1 \geq 0.$$

Then, Assumption 6 is satisfied. In particular, if r is a constant function, it reduces to the results discussed in the above sections.

There have been extensive researches work under the previous setting. Hence, we would like to devote more attention to the case when $\inf_{x \in \mathsf{E}} r(x) = 0$. As a main contribution in this direction, we managed to prove the existence and uniqueness of the viscosity solution to (4.5.2) in what follows.

Proposition 4.39. *Suppose that Assumption 6 holds. Let $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$, $f, g \in \mathcal{C}_b(\mathsf{E})$. There exists a unique viscosity solution to*

$$\min(rw - \mathcal{A}w - f, w - g) = 0. \quad (4.5.4)$$

Proof. Let us first observe that the viscosity solution to (4.5.4) is equivalent to the viscosity solution to

$$\min((1+r)w - \mathcal{A}w - (w+f), w-g) = 0. \tag{4.5.5}$$

Since $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$, it follows that the viscosity solution to (4.5.5) is equivalent to the viscosity solution associated with $(\frac{1}{1+r}\mathcal{A}, D(\mathcal{A}))$ to

$$\min(w - \frac{1}{1+r}\mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w) = 0, \tag{4.5.6}$$

where $\mathsf{F}w := \frac{w+f}{1+r}$ and $\mathsf{G}u := g$. Then, we only need to verify all the conditions of Assumption 2 and Assumption 3. The properties of G is obvious and we only prove the properties of F as follows.

(i) Since $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$ implies $\frac{1}{1+r} \in \mathcal{C}_b(\mathsf{E})$, combining with $f \in \mathcal{C}_b(\mathsf{E})$, (1) in Assumption 2 holds. Let $u_1, u_2 \in \mathcal{C}_b(\mathsf{E})$ and $0 \leq p \leq 1$, we have

$$\begin{aligned} p\mathsf{F}u_1 + (1-p)\mathsf{F}u_2 =& p\frac{u_1+f}{1+r} + (1-p)\frac{u_2+f}{1+r} \\ =& \frac{pu_1 + (1-p)u_2 + f}{1+r} \\ =& \mathsf{F}(pu_1 + (1-p)u_2). \end{aligned}$$

Thus, the operator F is convex. Additionally, if $u_1 \geq u_2$, $\mathsf{F}u_1 = \frac{u_1+f}{1+r} \geq \frac{u_2+f}{1+r} = \mathsf{F}u_2$. Therefore, Assumption 2 holds.

(ii) Using Assumption 6, let w_+ be the viscosity supersolution to $rw - \mathcal{A} - f = 0$. Define $w_+^* := w_+ + \|w_+\|_\infty + \|g\|_\infty$. Then, w_+^* is a viscosity supersolution to

$$\min(rw_+^* - \mathcal{A} - f, w - g) = 0,$$

which is equivalent with

$$\min(w - \frac{1}{1+r}\mathcal{A}w - \mathsf{F}w, w - g) = 0.$$

Therefore, by Corollary 4.12, (1) in Assumption 3 holds. Let $C > 0$, $p_1 = 1$ and $u \in \mathcal{C}_b(\mathsf{E})$. Since $\frac{1}{1+r} \leq 1$, $\mathsf{F}(u+C) - \mathsf{F}u = \frac{C}{1+r} \leq p_1 C$. Hence, Assumption 3 holds.

Therefore, by Theorem 4.10, the proof is finished. Here (1) in Assumption 3 follows from (4.5.3). Thus, there exists a unique viscosity solution to (4.5.6), which is equivalent to the viscosity solution to (4.5.4). By [Palczewski and Stettner, 2014, Theorem 1.1], the value function function is the viscosity solution to (4.5.4). ■

We should mention that there are no common results when the value function defined by (4.5.1) for any Feller process can be described as the unique viscosity solution to (4.5.3). They have to be studied case by case. However, it is possible to show that the viscosity solution to (4.5.3) always exists and is unique according to Proposition 4.39 under appropriate assumptions.

Let us look into more details with several examples satisfying Assumption 6. As emphasised above, we have to assume that the value function is the underlying viscosity solution, yet such assumption is justifiable in most cases.

4.5.1 Non-uniformly ergodic Markov process

Similarly as in [Palczewski and Stettner, 2014, Section 2.2], the authors introduced a zero potential function

$$q(x) = \lim_{T\to\infty} \boldsymbol{E}^x\left[\int_0^T (f(X(s)) - \mu(f))\mathrm{d}s\right], \tag{4.5.7}$$

where μ is an invariate measure of the process X and $\mu(f)$ is a negative constant dependent on f . By [Palczewski and Stettner, 2014, Lemma 2.2], the process $Z(t) = \int_0^t (f(X(s)) - \mu(f)) + q(X(t))$ is a martingale. Additionally, in this example, we assume that q is a bounded function and $\mu(f) < 0$. Then, let $0 < \kappa < -\mu(f)$, q is a viscosity supersolution to

$$-\mathcal{A}w - f + \kappa = 0.$$

However, the zero potential function q is not necessarily bounded from above if E is not compact. Thus, the value function in Palczewski and Stettner [2014] is only continuous but not bounded.

Corollary 4.40. *Assume that the conditions of [Palczewski and Stettner, 2014, Theorem 1.1] are in force and q is bounded and $\mu(f) < 0$. Then, the value function V defined by*

$$V(x) := \sup_\tau \boldsymbol{E}^x\left[\int_0^\tau f(X(s))\mathrm{d}s + g(X(\tau))\right],$$

is a continuous and bounded function. Additionally, the value function is the unique viscosity solution to

$$\min(-\mathcal{A}w - f, w - g) = 0$$

.

Proof. As mentioned before, there exists $\kappa < -\mu(f)$ such that the zeros potiential function $q(x)$ is the viscosity supersolution to

$$-\mathcal{A}w - f + \kappa = 0.$$

Then, Assumption 6 is satisfied and the claim follows from Proposition 4.39. ■

On the other hand, we should mention that we do not need the ergocity of $(\mathcal{G}, D(\mathcal{G}))$ to show there exists a unique viscosity solution to (4.5.4). For example, if there exists $C_0 < 0$ such that $f \leq C_0$, (4.5.3) in Corollary 4.39 holds for any Feller process.

4.5.2 Optimal stopping with random costs of observation

In this part, we are interested in

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^\tau f(X(s))\mathrm{d}s + g(X(\tau)) \right], \tag{4.5.8}$$

where X is a Feller process which does not necessarily satisfy the ergodic property. However, instead of the conditions required for a zero potential q, we impose certain conditions on the function f in the sequel.

Corollary 4.41. *Suppose that there exists a constant $c > 0$ such that $f \leq -c$. Then, the value function V defined by* (4.5.8) *is the unique viscosity solution to*

$$\min(-\mathcal{A}w - f, w - g) = 0.$$

Proof. Similar to the proof of Corollary 4.40. Choose $w_+ := 0$ and $\kappa = \frac{c}{2}$ and then Assumption 6 holds straightforward. ■

For example, let $f = -c$ be a constant function and $g \in \mathcal{C}_b(\mathsf{E})$. The value function of the optimal stopping time problem defined by

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \big[-c\tau + g(X(\tau)) \big]$$

can be characterized by the viscosity solution to

$$\min(c - \mathcal{A}w, w - g) = 0.$$

4.5.3 Finite time horizon optimal stopping problem

Finite time horizon optimal stopping problem is also a popular topic in previous literature. However, compared with infinite time horizon problems, such problems often do not include the discount cost, i.e., $a = 0$. Consequently, in this part, we will study the finite time horizon optimal stopping problems by Proposition 4.33 and obtain some direct results. Consider a process (D, X) on $\mathsf{E} := \mathbb{R}^+ \times \mathbb{R}^n$ with infinitesimal generator

$$D(\mathcal{A}^{(time)}) := \{u \in \mathcal{C}_*(\mathsf{E}); \frac{\partial}{\partial t}u(t,x) \in \mathcal{C}_0(\mathsf{E}), u_t \in D(\mathcal{A}) \text{ for } t \in \mathbb{R}^+\}, \quad (4.5.9)$$

$$\mathcal{A}^{(time)}u(t,x) := \frac{\partial}{\partial t}u(t,x) + b(t)\mathcal{A}u_t(x), \quad (4.5.10)$$

where $u_t(x) := u(t,x)$. By Mijatovic and Pistorius [2010], (D, Y) is a Feller process if $(\mathcal{A}, D(\mathcal{A})$ is the generator of the Feller semigroup. Additionally, let $T > 0$. We are interested in the following finite time horizon optimal stopping problem

$$V(d,x) := \boldsymbol{E}^{(d,x)}\left[\int_0^{\tau\wedge T} f(D(s), X(s))\mathrm{d}s + g(D(\tau\wedge T), X(\tau\wedge T))\right]. \quad (4.5.11)$$

Remark 4.42. *In particular, such optimal stopping problems are commonly studied for the time inhomogeneous diffusion, whose operator $\mathcal{A}^{(time)}$ is a parabolic operator (See for example Seydel [2009]). Here, we extend past results using Proposition 4.39 such that the operator is not restricted to a parabolic type.*

First, define the operator $(D(\mathcal{A}^{time}_{[0,T]}), \mathcal{A}^{time}_{[0,T]})$ by

$$D(\mathcal{A}^{(time)}_{[0,T]}) := \{u \in \mathcal{C}_b([0,T) \times \mathbb{R}^n); \text{there exists a continue extension } u_* \in D(\mathcal{A}^{time})\},$$

$$\mathcal{A}^{(time)}_{[0,T]}u(t,x) := \frac{\partial}{\partial t}u(t,x) + b(t)\mathcal{A}u_t(x).$$

Then, variational characterization of the value function is shown in the following corollary.

Corollary 4.43. *Assume that $f, g \in \mathcal{C}_b([0,T] \times \mathbb{R}^n)$, $f(T,x) = 0$ and $g(T,x) = 0$ for all $x \in \mathbb{R}^n$. Then, the value function V defined by (4.5.11) is in $\mathcal{C}_b([0,T] \times \mathbb{R}^n)$. Moreover, the value function V is the unique viscosity solution $w \in \mathcal{C}_b([0,T) \times \mathbb{R}^n)$ to*

$$\min(-\mathcal{A}^{(time)}_{[0,T]}w - f, w - g) = 0,$$

with the boundary condition $w(T, \cdot) = 0$.

Proof. Define the continuous extensions of the functions f and g by

$$\tilde{f}(t,x) := \begin{cases} f(t,x) & \text{for } x \in [0,T) \times \mathbb{R}^n \\ T-t & \text{for } x \in [T,T+1) \times \mathbb{R}^n \\ -1 & \text{for } x \in [T+1,\infty) \times \mathbb{R}^n \end{cases}$$

$$\tilde{g}(t,x) := \begin{cases} g(t,x) & \text{for } x \in [0,T) \times \mathbb{R}^n \\ 0 & \text{for } x \in [T,\infty) \times \mathbb{R}^n. \end{cases}$$

Due to the fact that $f(T,\cdot) = g(T,\cdot) = 0$, $\tilde{f}$ and $\tilde{g}$ are continuous functions. Define

$$\tilde{V}(x) := \sup_{\tau} \boldsymbol{E}^x \left[\int_0^{\tau} \tilde{f}(s,X(s))\mathrm{d}s + \tilde{g}(\tau, X(\tau)) \right].$$

Since $\tau \wedge T$ is also a F_t stopping time, compared with the value function defined by (4.5.11), we have $V \le \tilde{V}$. On the other hand, for any $\varepsilon > 0$, there exists a stopping time $\tilde{\tau}$ satisfying

$$\begin{aligned} \tilde{V}(x) - \varepsilon &\le \boldsymbol{E}^x \left[\int_0^{\tilde{\tau}} \tilde{f}(s,X(s))\mathrm{d}s + \tilde{g}(\tau, X(\tilde{\tau})) \right] \\ &= \boldsymbol{E}^x \left[\int_0^{\tilde{\tau}\wedge T} \tilde{f}(s,X(s))\mathrm{d}s + \mathbf{1}_{\tilde{\tau}>T} \int_{\tilde{\tau}\wedge T}^{\tilde{\tau}} \tilde{f}(s,X(s))\mathrm{d}s + \tilde{g}(\tilde{\tau}, X(\tilde{\tau})) \right] \\ &\le \boldsymbol{E}^x \left[\int_0^{\tilde{\tau}\wedge T} \tilde{f}(s,X(s))\mathrm{d}s + \tilde{g}(\tilde{\tau}\wedge T, X(\tilde{\tau}\wedge T)) \right], \end{aligned}$$

where the last inequality is from $f(t,x) \le 0$ and $g(t,x) = 0$ for $t \ge T$. Additionally, as $\varepsilon \to 0$, $\tilde{V} \ge V$. Therefore, the value function $\tilde{V}$ is equal to V. Since $\tilde{f}, \tilde{g} \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}^n)$, the value function $\tilde{V}$ in $\mathcal{C}_b(\mathsf{E})$ is a viscosity solution to

$$\min(-\mathcal{A}w - \tilde{f}, w - \tilde{g}) = 0. \tag{4.5.12}$$

Now, let us prove that the viscosity solution to (4.5.12) is unique. Define

$$u(t) := \begin{cases} -(\|f\|_\infty + 1) & \text{for } t \in [0,T+1) \\ -(\|f\|_\infty + 1) + (\|f\|_\infty + 1)(T+2-t) & \text{for } t \in [T+1,T+2) \\ 0. \end{cases}$$

Define $w_+(t,x) := \int_0^t u(s)\mathrm{d}s$. such that $\mathcal{A}^{(time)} w_+(t,x) = \frac{\partial w_+(t,x)}{\partial t} = u(t)$. As a result, we have

$$-\mathcal{A}^{(time)}w(t,x) - f(t,x) = -u(t) + f(t,x) \geq 1.$$

Hence, Assumption 6 holds. Then, by Proposition 4.39, there exists a unique viscosity solution to (4.5.12). Furthermore, since $\tilde{V}(t,x) = 0$ for $t \geq T$, then, the value function v can be characterized by the viscosity solution to

$$\min(-\mathcal{A}_0^{time}w - f, w - g) = 0$$

with boundary condition $w(T,x) = 0$. ■

Remark 4.44. *Compared with previous literature (see for example Seydel [2009]), we do not restrict the operator as a parabolic operator. However, we should mention that, the limitation of this method is that the functions f and g have to satisfy the boundary conditions $f(T,\cdot) = 0$ and $g(T,\cdot) = 0$. On the contrary, we suggest the condition $f(T,\cdot)$ vanishing at T can be omitted using a similar method as the proof of Theorem 3.26.*

4.5.4 Standard Brownian motion absorbed on both sides

Let Y be a Feller diffusion on state space $[-1,1]$ with absorbing boundaries at $y = -1, 1$, whose infinitesimal generator is

$$\begin{aligned} D(\mathcal{A}^{(a)}) &:= \{u \in \mathcal{C}_*^2([-1,1]); Du_{xx}(-1) = D_{xx}u(1) = 0\}; \\ \mathcal{A}^{(a)}u(x) &:= \frac{1}{2}D_{xx}u(x). \end{aligned} \tag{4.5.13}$$

Our optimal stopping time problem is defined by

$$V_Y(y) := \sup_{\tau} \boldsymbol{E}^y\left[\int_0^\tau f(Y(s))\mathrm{d}s + g(Y(\tau))\right].$$

Suppose that $f(-1) \leq 0$ and $f(1) \leq 0$.

We analogously assume that the value function V_Y is the viscosity solution to

$$\min(-\mathcal{A}w - f, w - g) = 0.$$

Using Proposition 4.39, the following corollary shows the uniqueness of this viscosity solution.

Corollary 4.45. *Suppose that $f, g \in \mathcal{C}_b(\mathsf{E})$ with $f(-1) < 0$ and $f(1) < 0$. There* *exists a unique viscosity solution to*

$$\min(-\mathcal{A}^{(a)}w - f, w - g) = 0,$$

where $(\mathcal{A}^{(a)}, D(\mathcal{A}^{(a)}))$ *is defined by* (4.5.13)

Proof. Here, we only need to verify Assumption 6 holds true. Due to $f(-1), f(1) < 0$, write a constant $C_0 = \min(-f(-1), -f(1))$ such that $C_0 > 0$. Define $w_+ := -\frac{\|f\|_\infty+1}{2}x^2 + \|f\|_\infty + 1$ and $\kappa = \min(C_0, 1)$. We want to show that w_+ is a viscosity supersolution to

$$-\mathcal{A}w - f - \kappa = 0$$

First, we show the viscosity property for $x \in (-1, 1)$. Since $-\mathcal{A}w_+(x) = -D_{xx}w_+(x) = \|f\|_\infty + 1$, we have $-\mathcal{A}w - f - \kappa = \|f\|_\infty + 1 - f - \min(C_0, 1) \geq 0$. For the case $x = -1$, let $\phi \in D(\mathcal{A})$. Because $D_{xx}\phi(-1) = 0$, we have $-\mathcal{A}\phi - f - \kappa = -f(-1) - \kappa \geq -f(-1) - \min(-f(-1), -f(1)) \geq 0$. The viscosity property of w_+ at $x = 1$ can be proved in a similar way. Therefore, w_+ is a viscosity supersolution such that Assumption 6 is satisfied. ■

Remark 4.46. *We want to mention that* $f(-1), f(1) < 0$ *is a necessary condition to guarantee the uniqueness of the viscosity solution. Otherwise, the uniqueness of the viscosity solution will be challenged. For example, let* f *be a constant function such that* $f = 0$*. Then, any constant function* w *satisfying* $w \geq \|g\|_\infty$ *is a viscosity solution to*

$$\min(-\mathcal{A}w - f, w - g) = 0.$$

Chapter 5

Optimal Stopping Problems for Multiplicative Functional

The detail preliminaries in this section can be found in Section 2.3.

5.1 Problem formulation

In this chapter, let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ be a Markov process with state space E_∂ and M be a multiplicative functional of X. Recall that we impose the following conditions:

1. ∂ is an absorbing state such that $X(t) = \partial$ for any $t \geq s$ if $X(s) = \partial$,
2. there is a distinguished point w_∂ in Ω such that $X(0)(\omega_\partial) = \partial$.
3. the life time of X is defined by $\eta_X := \inf\{t \geq 0; X(t) = \partial\}$.
4. the time horizon is extended to $\bar{\mathbb{R}}_+ := [0, \infty]$ such that $X_\infty(\omega) = \partial$ and $\theta_\infty(\omega) = \omega_\partial$ for all $\omega = \Omega$.

Let $\mathcal{T}$ be a family of all $\mathcal{F}_t$-stopping times. Let f and g be two real-valued Borel measurable functions on E. The optimal stopping problem is to find the optimal stopping time $\tau^* \in \mathcal{T}$ which maximizes the following objective function

$$J_X^M(x, \tau) := \boldsymbol{E}^x \left[\int_0^\tau e^{-as} M(s) f(X(s))\, \mathrm{d}s + e^{-a\tau} M(\tau) g(X(\tau)) \right] \text{ for } x \in \mathsf{E}_\partial \text{ and } \tau \in \mathcal{T}, \tag{5.1.1}$$

where $a > 0$ is a constant discount factor. Then, its value function is defined by

$$V_X^M(x) := \sup_{\tau \in \mathcal{T}} J_X^M(x, \tau) = J_X^M(x, \tau^*) \text{ for } x \in \mathsf{E}_\partial. \tag{5.1.2}$$

5.1.1 Examples

We will follow the examples in Example 2.21 and present some standard optimal stopping problems with multiplicative functionals. The multiplicative functional can used to extend the orignal infinite optimal stopping problem. We first give several examples about multiplicative functional.

Example 5.1. *$M(t) = e^{-\delta t}$ is a multiplicative functional for $\delta \geq 0$. The corresponding value function is given by*

$$V(x) = \sup_{\tau \in \mathcal{T}} \boldsymbol{E}^x \left[\int_0^\tau e^{-(a+\delta)s} f(X(s)) \, \mathrm{d}s + e^{-(a+\delta)\tau} g(X(\tau)) \right]. \tag{5.1.3}$$

More generally, the discount rate can be dependent on the state as follows.

Example 5.2. *Let r be a positive continuous bounded function and $M(t) = exp\left(-\int_0^t (r(X(s)))\mathrm{d}s\right)$ for $t \geq 0$. Then, the value function is*

$$V(x) = \sup_{\tau \in \mathcal{T}} \boldsymbol{E}^x \left[\int_0^\tau e^{-\int_0^s (r(X(z))+a)\mathrm{d}z} f(X(s)) \, \mathrm{d}s + e^{-\int_0^\tau (r(X(z))+a)\mathrm{d}z} g(X(\tau)) \right]. \tag{5.1.4}$$

The last example is on the optimal stopping time before hitting a subset $\mathcal{O}$ of E.

Example 5.3. *Let $\mathcal{O}$ be an open subset and a $\mathcal{F}_t$-stopping time $\tau_{\mathcal{O}} := \inf\{t \geq 0; X(t) \notin \mathcal{O}\}$. Let $M(t)(\omega) = \mathbf{1}_{t \in [0, \tau_{\mathcal{O}}(\omega))}$ for $\omega \in \Omega$. Then, the value function is*

$$V(x) = \sup_{\tau \in \mathcal{T}} \boldsymbol{E}^x \left[\int_0^{\tau \wedge \tau_{\mathcal{O}}} e^{-as} f(X(s)) \, \mathrm{d}s + e^{-a\tau} g(X(\tau)) \mathbf{1}_{\tau < \tau_{\mathcal{O}}} \right]. \tag{5.1.5}$$

Remark 5.4. *More specifically, we can define a finite time horizon optimal stopping time problem. Consider a time inhomogeneous Feller process (D, X) defined in (4.5.9). Define a multiplicative functional $M(t)(\omega) = \mathbf{1}_{t \in [0,T)}$. Then, the value function is*

$$V(x) = \sup_{\tau \in \mathcal{T}} \boldsymbol{E}^x \left[\int_0^{\tau \wedge T} e^{-as} f(X(s)) \, \mathrm{d}s + e^{-a\tau} g(X(\tau)) \mathbf{1}_{\tau < T} \right]. \tag{5.1.6}$$

5.1.2 Assumptions

Define an operator $\{\mathcal{Q}_t\}_{t \geq 0}$ on $B(\mathsf{E}_\partial)$ by

$$\mathcal{Q}_t w(x) := \boldsymbol{E}^x[M(t) w(X(t)); X(t) \in \mathsf{E}] \text{ for } x \in \mathsf{E}_\partial \text{ and } t \geq 0. \tag{5.1.7}$$

To simplify our problem, without loss of generality, in this chapter, we always suppose that

1. X is a normal, cádlág, quasi-left-continuous and strong Markov process,
2. $\mathcal{F}$ and $\mathcal{F}_t$ are completed and right continuous filtrations,
3. M is a right continuous and non-increasing multiplicative functional.
4. $M(t)(\omega) = 0$ if $X(t)(\omega) = \partial$.

Recall that a point $x \in \mathsf{E}$ is called a *permanent point* for M if $\boldsymbol{P}^x(M(0) = 1) = 1$ and define E_M by the set of all the permanent points. We furthermore impose the following assumptions:

Assumption 7. *$\mathsf{E}_M \in \mathcal{E}$, M is $\mathcal{F}^0$-measurable and M is a strong multiplicative functional, where $\mathcal{F}^0$ is the natural filtration of the process X.*

Assumption 8.

1. *$f|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$ and $g|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$.*
2. *$\mathcal{Q}_t w|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$ for $w \in B(\mathsf{E}_\partial)$ satisfying $w|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$.*

Remark 5.5. *Assumption 7 is related with some fundamental properties of the multiplicative functional. It is necessary condition when we reduce our optimal stopping problems with multiplicative functionals to the standard one (see Section 5.2). On the other hand, to prove the viscosity property, Assumption 8 shows the Feller property of the semigroup $\{\mathcal{Q}_t\}_{t\geq 0}$.*

5.2 Equivalent Problems

It is not straightforward to solve the optimal stopping problems for general Markov process X with general multiplicative functional M. Hence we first need to transform the optimal stopping problem for Markov process X into two equivalent optimal stopping problem for Markov processes $\hat{X}$ and $\tilde{X}$, where $\tilde{X}$ is normal Markov process. In addition, using Assumption 8, the process $\tilde{X}$ is a Feller process and hence one can apply results of viscosity solution in Chapter 3 (e.g. Theorem 3.17 and Theorem 3.9) to solve the optimal stopping problem (5.1.3). We start by giving several preliminary results that enable the transformation of the Markov process X with multiplicative functional M. We use the following definition of a Markov process (see [Blumenthal and Getoor, 2007, Chapter 1, Section 3]):

Definition 5.6. $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \boldsymbol{P}^x)$ *is called a Markov process with state space* $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ *provided that*

1. *For each* $t \in \mathbb{R}^+$, $X(t)$ *is* $\mathcal{F}_t$*-measurable.*

2. *For each* $t \in \mathbb{R}^+$ *and* $B \in \mathcal{E}$, *the map* $x \mapsto \boldsymbol{P}^x(X(t) \in B)$ *from* E *to* $[0,1]$ *is* $\mathcal{E}$*-measurable.*

3. *For all* $t, h \in \bar{\mathbb{R}}^+$, $X(t) \circ \theta_h = X(t+h)$.

4. *For all* $x \in \mathsf{E}_\partial$, $B \in \mathcal{E}_\partial$ *and* $t, s \in \bar{\mathbb{R}}^+$,

$$\boldsymbol{P}^x(X(t+s) \in B | \mathcal{F}_t) = \boldsymbol{P}^{X(t)}(X(s) \in B). \tag{5.2.1}$$

5. $\boldsymbol{P}^\partial(X(0) = \partial) = 1$.

We will construct two Markov processes $\hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\boldsymbol{P}}^x)$ with state space $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ and $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{\boldsymbol{P}}^x)$ with state space $(\mathsf{E}_\partial^M, \mathcal{E}_\partial^M)$, where

$$\mathsf{E}_\partial^M := \mathsf{E}_M \cup \{\partial\}$$

and $\mathcal{E}_\partial^M$ is the trace of E_∂ restricted on E_M. As shown in Theorem 5.12, their value function defined by (5.1.3) coincides with the value functions defined by

$$V_{\hat{X}}(x) := \sup_{\tau \in \hat{\mathfrak{T}}} J_{\hat{X}}(x, \tau) = J_{\hat{X}}(x, \hat{\tau}^*) \text{ for } x \in \mathsf{E}_\partial, \tag{5.2.2}$$

$$V_{\tilde{X}}(x) := \sup_{\tau \in \hat{\mathfrak{T}}} J_{\tilde{X}}(x, \tau) = J_{\tilde{X}}(x, \tilde{\tau}^*) \text{ for } x \in \mathsf{E}_\partial^M, \tag{5.2.3}$$

where $\hat{\mathfrak{T}}$ is the set of all $\hat{\mathcal{F}}_t$-stopping time, $\tilde{\mathfrak{T}}$ is the set of all $\tilde{\mathcal{F}}_t$-stopping time and

$$J_{\hat{X}}(x, \tau) := \hat{\boldsymbol{E}}^x \left[\int_0^\tau e^{-as} f(\hat{X}(s)) \mathrm{d}s + e^{-a\tau} g(\hat{X}(\tau)) \right], \tag{5.2.4}$$

$$J_{\tilde{X}}(x, \tau) := \tilde{\boldsymbol{E}}^x \left[\int_0^\tau e^{-as} f(\tilde{X}(s)) \mathrm{d}s + e^{-a\tau} g(\tilde{X}(\tau)) \right]. \tag{5.2.5}$$

Next we construct the two processes $\hat{X}$ and $\tilde{X}$.

5.2.1 Process Transformation to $\hat{X}$

Let $M = \{M(t)\}_{t \geq 0}$ be a right continuous and non-increasing multiplicative functional of X defined by Definition 2.17. Next, we use M to construct a new Markov process $\hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\boldsymbol{P}}^x)$ with state space $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ defined as follows. (See also [Blumenthal and Getoor, 2007, Section III.3])

1. Let $\hat{\Omega} := \Omega \times \bar{\mathbb{R}}_+$ and write $\hat{\Omega}_t := \Omega \times (t, \infty]$. Define two natural projections $\pi : \hat{\Omega} \to \Omega$ and $\gamma : \hat{\Omega} \to \bar{\mathbb{R}}_+$ such that $\pi(\hat{\omega}) = \omega$ and $\gamma(\hat{\omega}) = \lambda$ for $\hat{\omega} = (\omega, \lambda) \in \Omega$. Furthermore, $\hat{\omega}_\partial := (\omega_\partial, 0)$.

2. $\hat{\mathcal{F}} := \mathcal{F} \times \bar{\mathcal{R}}_+$, where $\mathcal{R}$ is σ-algebra of Borel subsets of $\bar{\mathbb{R}}_+$.

3. $\hat{\mathcal{F}}_t = \{\hat{\Lambda} \subseteq \hat{\Omega}; \quad \hat{\Lambda} \cap \hat{\Omega}_t = \Lambda \times (t, \infty] \text{ for } \Lambda \in \mathcal{F}_t\}$,

4. $\hat{X}(t)(\hat{\omega}) := X(t)(w)$ for $t < \lambda$ and $\hat{X}(t)(\hat{\omega}) := \partial$ for $t \geq \lambda$ given $\hat{\omega} = (\omega, \lambda) \in \hat{\Omega}$.

5. $\hat{\theta}_t\hat{\omega} := (\theta_t\omega, (\lambda - t) \vee 0)$ given $\hat{\omega} = (\omega, \lambda) \in \hat{\Omega}$. Here, we take $\infty - \infty = 0$ such that $\hat{\theta}_\infty\hat{\omega} = \hat{\omega}_\partial = (\omega_\partial, 0)$.

6. Let α_w be a measure on $[0, \infty]$ for each $w \in \Omega$ by setting $\alpha_\omega(\{0\}) := 0$ and $\alpha_\omega((t, \infty]) := M(t)(\omega)$ for $t \in [0, \infty]$. Let $\hat{\Lambda} \in \hat{\mathcal{F}}$ and $\hat{\Lambda}^\omega := \{\lambda; (\omega, \lambda) \in \hat{\Lambda}\}$. Note that $\hat{\Lambda}^\omega \in \bar{\mathbb{R}}_+$ and then the mapping $\omega \to \alpha_\omega(\hat{\Lambda}^\omega)$ is $\mathcal{F}$-measurable. Thus, the probability measure $\hat{\boldsymbol{P}}^x$ can be defined by

$$\hat{\boldsymbol{P}}^x(\hat{\Lambda}) := E^x\big[\alpha_w(\hat{\Lambda}^w)\big] \text{ for } x \in \mathsf{E}_M,$$
$$\hat{\boldsymbol{P}}^x(\hat{\Lambda}) := \mathbf{1}(\hat{\omega}_\partial \in \hat{\Lambda}) \text{ for } x \in \mathsf{E}_\partial \setminus \mathsf{E}_M,$$

where $\mathbf{1}$ is the indicator function.

Define

$$B_0(\mathsf{E}_\partial) := \{w \in B(\mathsf{E}_\partial); w(\partial) = 0\}.$$

There are two useful results from [Blumenthal and Getoor, 2007, Theorem III.3.4] and [Blumenthal and Getoor, 2007, Theorem III.3.9].

Theorem 5.7. *Suppose that Assumption 7 holds.* $\hat{X}$ *is a Markov process with state space* $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ *satisfying*

$$\hat{\boldsymbol{E}}^x\big[w(\hat{X}(t))\big] = \boldsymbol{E}^x\big[w(X(t))M(t)\big] \text{ for } x \in \mathsf{E} \text{ and } w \in B_0(\mathsf{E}_\partial). \tag{5.2.6}$$

Proof. This theorem follows from [Blumenthal and Getoor, 2007, Theorem III.3.4]. However, [Blumenthal and Getoor, 2007, Theorem III.3.4] use $(\mathsf{E}_\partial, \mathcal{E}_\partial^*)$ as the state space. In fact, one can show that the conclusion of [Blumenthal and Getoor, 2007, Theorem III.3.4] still holds when $\mathcal{E}_\partial^*$ is replaced by $\mathcal{E}_\partial$.

We will show that $\hat{X}$ with state space $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ satisfies Definition 5.6.

1. Since $\mathcal{E}_\partial \subseteq \mathcal{E}_\partial^*$, $X(t)$ is $\mathcal{F}_t/\mathcal{E}_\partial^*$ implies that $X(t)$ is also $\mathcal{F}_t/\mathcal{E}_\partial$.

2. Let $B \in \mathcal{E}_\partial$. Since $\hat{\boldsymbol{P}}^x(\hat{X}(t) \in B) = \boldsymbol{E}^x(\mathbf{1}_{X(t)\in B}M(t))$, $\mathbf{1}_{X(t)\in B}$ and $M(t)$ are $\mathcal{F}^0$-measurable. Thus $x \to \hat{\boldsymbol{P}}^x(\hat{X}(t) \in B)$ is $\mathcal{E}_\partial$-measurable.

3. Since $\mathcal{E}_\partial \subseteq \mathcal{E}_\partial^*$ and $B(\mathsf{E}_\partial) \subseteq B^*(\mathsf{E}_\partial)$, other properties in Definition 5.6 follow.

This completes the proof. ■

We will also need the following result which is from [Blumenthal and Getoor, 2007, Theorem III.3.8].

Theorem 5.8. *Let $\hat{\tau} \in \hat{\mathcal{T}}$. Then, there exists a unique $\tau \in \mathcal{T}$ such that*

$$\hat{\tau}(\hat{\omega}) \wedge \gamma(\hat{\omega}) = \tau(\pi(\hat{\omega})) \wedge \gamma(\hat{\omega}) \text{ for } \hat{\omega} \in \hat{\Omega}. \tag{5.2.7}$$

Furthermore, for any $\hat{\Lambda} \in \hat{\mathcal{F}}_{\hat{\tau}}$, there exists a set $\Lambda \in \mathcal{F}_\tau$ such that

$$\hat{\Lambda} \cap \{\hat{\omega} \in \hat{\Omega}; \hat{\tau}(\hat{\omega}) \leq \gamma(\hat{\omega})\} = (\Lambda \times [0, \infty]) \cap \{\hat{\omega} \in \hat{\Omega}; \tau(\pi(\hat{\omega})) \leq \gamma(\hat{\omega})\}. \tag{5.2.8}$$

Corollary 5.9. *Let $\hat{\tau} \in \hat{\mathcal{T}}$ and $\tau \in \mathcal{T}$ satisfying (5.2.7). Then, for $w \in B_0(\mathsf{E}_\partial)$*

$$\boldsymbol{E}^x\Big[e^{-a\tau} M(\tau) w(X(\tau))\Big] = \hat{\boldsymbol{E}}^x\Big[e^{-a\hat{\tau}} w(\hat{X}(\hat{\tau}))\Big], \tag{5.2.9}$$

$$\boldsymbol{E}^x\Big[\int_0^\tau e^{-as} M(s) w(X(s)) \mathrm{d}s\Big] = \hat{\boldsymbol{E}}^x\Big[\int_0^{\hat{\tau}} e^{-as} w(\hat{X}(s)) \mathrm{d}s\Big]. \tag{5.2.10}$$

Proof. First let $x \in \mathsf{E}_\partial \setminus \mathsf{E}_M$ Since $\hat{\omega}_\partial = (\omega_\partial, 0)$ and $\hat{X}(t)(\hat{\omega}) = \partial$ for $t \geq \lambda$ given $\hat{\omega} = (\omega, \lambda)$, it follows that $\hat{X}(t)(\hat{\omega}_\partial) = \partial$ for $t \geq 0$. In addition $\hat{\tau}$ is positive, hence $\hat{X}(\hat{\tau})(\hat{\omega}_\partial) = \partial$. Therefore,

$$\hat{\omega}_\partial \in \hat{\Lambda} := \{\hat{\omega} \in \hat{\Omega} \text{ s.t. } \hat{X}(\hat{\tau})(\hat{\omega}) = \partial\}$$

and then $\boldsymbol{P}^x(\hat{\Lambda}) = 1$ for $x \in \mathsf{E}_\partial \setminus \mathsf{E}_M$ by the definition of $\hat{X}$. Hence

$$\hat{\boldsymbol{E}}^x[e^{-a\hat{\tau}} w(\hat{X}(\hat{\tau}))] = \hat{\boldsymbol{E}}^x[e^{-a\hat{\tau}} w(\partial)] = 0.$$

On the other hand, let $x \in \mathsf{E}_M$. Since $\hat{X}(t)(\hat{\omega}) = \partial$ for all $t \geq \gamma(\hat{\omega})$ and $w(\partial) = 0$, we have

$$\hat{\boldsymbol{E}}^x\Big[e^{-a\hat{\tau}} w(\hat{X}(\hat{\tau}))\Big] = \hat{\boldsymbol{E}}^x\Big[e^{-a\hat{\tau}} w(\hat{X}(\hat{\tau})) \mathbf{1}_{\hat{\tau} < \gamma}\Big].$$

Thus, by (5.2.7),

$$\begin{aligned}\hat{\boldsymbol{E}}^{x}\Big[e^{-a\hat{\tau}}w(\hat{X}(\hat{\tau}))\mathbf{1}_{\hat{\tau}<\gamma}\Big] &= \hat{\boldsymbol{E}}^{x}\Big[e^{-a\tau(\pi(\hat{\omega}))}w(X(\tau)(\pi(\hat{\omega})))\mathbf{1}_{\tau(\pi(\hat{\omega}))<\gamma(\hat{\omega})}\Big],\\ &= \boldsymbol{E}^{x}\Big[e^{-a\tau(\omega)}w(X(\tau)(\omega))\alpha_{\omega}((\tau(\omega),\infty])\Big]\\ &= \boldsymbol{E}^{x}\Big[e^{-a\tau}M(\tau)w(X(\tau))\Big].\end{aligned}$$

Then, (5.2.9) is proved. Similarly, (5.2.10) can be proved by

$$\begin{aligned}\hat{\boldsymbol{E}}^{x}\Big[\int_0^{\hat{\tau}} e^{-as}w(\hat{X}(s))\mathrm{d}s\Big] &= \hat{\boldsymbol{E}}^{x}\Big[\int_0^{\hat{\tau}} e^{-as}w(\hat{X}(s))\mathbf{1}_{s<\gamma}\mathrm{d}s\Big]\\ &= \hat{\boldsymbol{E}}^{x}\Big[\int_0^{\tau(\pi(\hat{\omega}))} e^{-as}w(X(s)(\pi(\hat{\omega})))\mathbf{1}_{s<\gamma(\hat{\omega})}\mathrm{d}s\Big]\\ &= \boldsymbol{E}^{x}\Big[\int_{\mathbb{R}+}\Big(\int_0^{\tau(\omega)} e^{-as}f(X(s)(\omega))\mathbf{1}_{s<\lambda}\mathrm{d}s\Big)\alpha_{\omega}(d\lambda)\Big]\\ &= \boldsymbol{E}^{x}\left[\int_0^{\tau} e^{-as}f(X(s))M(s)\mathrm{d}s\right].\end{aligned}$$

The proof is completed. ■

5.2.2 Process transformation to $\tilde{X}$

One drawback of the process $\hat{X}$ is that it is not necessary to be a normal Markov process, since $\hat{\boldsymbol{P}}^{x}(X(0)=x)=0$ for all $\mathsf{E}\setminus\mathsf{E}_M$. However, after imposing several suitable conditions, we may overcome this by restricting $\hat{X}$ on the state space $\mathsf{E}_\partial^M := \mathsf{E}_M\cup\{\partial\}$. More precisely, we define the restriction of a Markov process $\tilde{X} = (\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathfrak{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{P}^x)$ on E_∂^M as follows.

1. $\tilde{\Omega} := \{\tilde{\omega}\in\hat{\Omega}; \hat{X}(t)(\tilde{\omega})\in\mathsf{E}_\partial^M \text{ for all } t\geq 0\}$.
2. $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{F}}_t$ are defined as the trace of $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{F}}_t$ on $\tilde{\Omega}$, respectively,
3. Given $\tilde{\omega}\in\tilde{\Omega}$, $\tilde{X}(t)(\tilde{\omega}) := \hat{X}(t)(\hat{\omega})$ and $\tilde{\theta}_t\tilde{\omega} := \hat{\theta}_t\hat{\omega}$ for all $t\geq 0$.
4. For each $x\in\mathsf{E}_\partial^M$, $\tilde{\boldsymbol{P}}^{x}$ is the trace of $\hat{\boldsymbol{P}}^{x}$ on $(\tilde{\Omega},\tilde{\mathfrak{F}})$.

It can be seen that one necessary condition to show $\tilde{X}$ is a Markov process is $\tilde{\boldsymbol{P}}^{x}(\tilde{\Omega}) = 1$.

Theorem 5.10. *Suppose that Assumption 7 holds. $\tilde{X}$ is a normal Markov process with state space $(\mathsf{E}_\partial^M, \mathcal{E}_\partial^M)$ and satisfies*

$$\tilde{\boldsymbol{E}}^{x}[h(\tilde{X}(t))] = \hat{\boldsymbol{E}}^{x}[h(\hat{X}(t))] \text{ for } h\in B_0(\mathsf{E}_\partial). \tag{5.2.11}$$

Proof. Let $x \in \mathsf{E}_\partial^M$. We first prove that $\tilde{\boldsymbol{P}}^x$ is a probability measure, that is,

$$\tilde{\boldsymbol{P}}^x(\tilde{\Omega}) = \hat{\boldsymbol{P}}^x(\hat{X}(t) \in \mathsf{E}_M^\partial \text{ for all } t \geq 0) = 1. \tag{5.2.12}$$

Since $\tau_{\mathsf{E}_M} \in \mathcal{T}$ by Assumption 7, $\hat{\tau}_{\mathsf{E}_M} := \inf\{t \geq 0; \hat{X}(t) \notin \mathsf{E}_M\}$ is a $\hat{\mathfrak{F}}_t$-stopping time by the fact that $\{\hat{\tau} \leq t\} \cap (\Omega \times (t, \infty]) = \{\tau \leq t\} \times (t, \infty]$. Then, since (5.2.12) is true when $x = \partial$, it is sufficient to prove $\hat{\boldsymbol{P}}^x(\hat{\tau}_{\mathsf{E}_M} < \gamma) = 0$ for all $x \in \mathsf{E}_M$. For $x \in \mathsf{E}_M$, $\hat{\boldsymbol{P}}^x(\hat{\tau}_{\mathsf{E}_M} < \gamma) = \boldsymbol{E}^x[\alpha_\omega((\tau_{\mathsf{E}_M}(\omega), \infty])] = \boldsymbol{E}^x[M(\tau_{\mathsf{E}_M})]$. By [Blumenthal and Getoor, 2007, Proposition III.4.22], since M is a regular multiplicative functional, we have $M(\tau_{\mathsf{E}_M}) = 0$ almost surely. Then, (5.2.12) holds for all $x \in \mathsf{E}_M^\partial$ and $\tilde{\boldsymbol{P}}^x$ is a probability measure.

Secondly, it is obvious that $\tilde{X}(t)(\tilde{\theta}_s(\tilde{\omega})) = \tilde{X}_{t+s}(\tilde{\omega})$, for all $t, s \geq 0$ and $\tilde{\omega} \in \tilde{\Omega}$. Moreover, the we claim that random variable $\tilde{X}(t)$ is $\tilde{\mathfrak{F}}_t$-measurable. Indeed, let $\tilde{A} \in \mathcal{E}_M^\partial$ and there exists a $A \in \mathcal{E}_\partial$ such that $A \cup \mathsf{E}_M^\partial = \tilde{A}$. Then, for $t \geq 0$, $\{\tilde{X}(t) \in \tilde{A}\} = \{\hat{X}(t) \in A\} \cap \tilde{\Omega}$. Since $\{\hat{X}(t) \in A\} \in \hat{\mathfrak{F}}_t$, we have $\tilde{X}(t)$ is $\tilde{\mathfrak{F}}_t$ measurable. Therefore, it is sufficient to prove that $\tilde{X}$ is a Markov process by showing that for each $x \in \mathsf{E}_M^\partial$, $s, t \geq 0$, $\tilde{A} \in \tilde{\mathfrak{F}}_s$ and $w \in B(\mathsf{E}_M^\partial)$,

$$\tilde{\boldsymbol{E}}^x[w(\tilde{X}(s+t)); \tilde{A}] = \tilde{\boldsymbol{E}}^x[\tilde{\boldsymbol{E}}^{\tilde{X}(s)}[w(\tilde{X}(t))]; \tilde{A}].$$

Since there exists $\hat{A} \in \hat{\mathfrak{F}}_s$ such that $\hat{A} \cap \tilde{\Omega} = \tilde{A}$ and $\tilde{\Omega}^c$ is a null space by (5.2.12) under $\hat{\boldsymbol{P}}^x$ for all $x \in \mathsf{E}_M^\partial$, we have $\tilde{\boldsymbol{E}}^x[w(\tilde{X}(s+t)); \tilde{A}] = \hat{\boldsymbol{E}}^x[\hat{w}(\hat{X}(s+t)); \hat{A}] = \hat{\boldsymbol{E}}^x[\hat{\boldsymbol{E}}^{\hat{X}(s)}[\hat{w}(\hat{X}(t))]; \hat{A}] = \tilde{\boldsymbol{E}}^x[\tilde{\boldsymbol{E}}^{\tilde{X}(s)}[w(\tilde{X}(t))]; \tilde{A}]$, where $\hat{w}$ is the extension of w to E_∂ vanishing in $\mathsf{E} \setminus \mathsf{E}_M$.

Thirdly, we prove that the Markov process $\tilde{X}$ is normal. Since E_M is a set of all the permanent points i.e., $\boldsymbol{P}^x(M(0) = 1) = 1$ for $x \in \mathsf{E}_M$, then $\hat{\boldsymbol{P}}^x(X(0) = x) = \boldsymbol{P}^x(M(0) = 1) = 1$ for all $x \in \mathsf{E}$. Additionally, $\hat{\boldsymbol{P}}^\partial(X(0) = \partial) = 1$ by Definition 5.6. Hence, $\tilde{\boldsymbol{P}}^x(X(0) = x) = \hat{\boldsymbol{P}}^x(X(0) = x)$ implies $\tilde{X}$ is normal. ■

Furthermore, we also have the following theorem concerning the stopping time.

Theorem 5.11. *Suppose that Assumption 7 holds. For any $\tilde{\mathfrak{F}}_t$-stopping time $\tilde{\tau}$, there exists a $\hat{\mathfrak{F}}_t$-stopping time $\hat{\tau}$ such that $\tilde{\tau} = \hat{\tau}$ on $\tilde{\Omega}$. In addition, for each $\tilde{\Lambda} \in \tilde{\mathfrak{F}}_\tau$, there exists $\hat{\Lambda} \in \hat{\mathfrak{F}}_\tau$ with $\tilde{\Lambda} = \hat{\Lambda} \cap \tilde{\Omega}$. Furthermore, if $\hat{X}$ is a strong Markov process, $\tilde{X}$ is also a strong Markov process.*

Proof. For $a \geq 0$, let $\tilde{A}_a \in \tilde{\mathfrak{F}}_a$ and define a $\tilde{\mathfrak{F}}_t$-stopping time $\tilde{\tau}_a$ by

$$\tilde{\tau}_a := a, \text{ for } \tilde{\omega} \in \tilde{A} \quad \text{otherwise } \tilde{\tau}_a := \infty.$$

Since there exists $\hat{A}_a \in \hat{\mathfrak{F}}_a$ such that $\hat{A}_a \cap \tilde{\Omega} = \tilde{A}$, we define a $\hat{\mathfrak{F}}_t$-stopping time $\hat{\tau}_a$

of the same form as the above by replacing $\tilde{A}$ by $\hat{A}$. Hence, $\tilde{\tau}_a = \hat{\tau}_a$ on $\tilde{\Omega}$. Since every $\tilde{\mathfrak{F}}_t$-stopping time $\tilde{\tau}$ can be constructed by $\tilde{\tau} = \inf_{n\in\mathbb{N}^+} \tilde{\tau}_{a_n}$, we can find a corresponding $\hat{\mathfrak{F}}_t$-stopping time $\hat{\tau} = \inf_{n\in\mathbb{N}^+} \hat{\tau}_{a_n}$ such that $\tilde{\tau} = \hat{\tau}$ on $\tilde{\Omega}$.

Moreover, for each $\tilde{\Lambda} \in \tilde{\mathfrak{F}}_{\tilde{\tau}}$, we can define

$$\tilde{\Lambda} = \cup_{a\in\mathbb{Q}^+\cup\infty}\tilde{\Lambda} \cap \{\tilde{\tau} \le a\}.$$

Since for $a \in \mathbb{Q}^+ \cup \infty$, $\tilde{\Lambda} \cap \{\tilde{\tau} \le a\} \in \tilde{\mathfrak{F}}_a$, then there exists $\hat{\Lambda}_a \in \hat{\mathfrak{F}}_a$ such that $\hat{\Lambda}_a \cap \tilde{\Omega} = \tilde{\Lambda} \cap \{\tilde{\tau} \le a\}$. We can define $\hat{\Lambda} = \cup_{a\in\mathbb{Q}^+\cup\infty}\hat{A}_a$, which is in $\hat{\mathfrak{F}}$ and $\hat{\Lambda} \cap \tilde{\Omega} = \tilde{\Lambda}$. Then, for $s \ge 0$, $\hat{\Lambda} \cap \{\hat{\tau} \le s\} = \cup_{a\in[0,s]\cap(\mathbb{Q}^+\cup\infty)}\hat{A}_a \in \hat{\mathfrak{F}}_s$. Therefore, $\hat{\Lambda} \in \hat{\mathfrak{F}}_{\hat{\tau}}$.

Now, we prove that $\tilde{X}$ is a strong Markov process. Let $\tilde{B} \in \mathcal{E}_M^\partial$ and there exists $\hat{B} \in \mathcal{E}_\partial$ such that $\hat{B} \cap \mathsf{E}_M = \tilde{B}$. Then, for $t \ge 0$, $\left\{\tilde{X} \in \tilde{B}\right\} = \{\hat{X} \in \hat{B}\} \cap \tilde{\Omega} \in \tilde{\mathfrak{F}}_t$. Therefore, strong Markov process $\tilde{X}$ is proved by for $x \in \mathsf{E}_M^\partial$, $t \ge 0$ and $w \in B(\mathsf{E}_M^\partial)$, $\tilde{\boldsymbol{E}}^x\left[w(\tilde{X}(t+\tilde{\tau}))\right] = \hat{\boldsymbol{E}}^x\left[\hat{w}(\hat{X}(t+\hat{\tau}))\right] = \hat{\boldsymbol{E}}^x\left[\hat{\boldsymbol{E}}^{\hat{X}(\hat{\tau})}\left[\hat{w}(\hat{X}(t))\right]\right] = \tilde{\boldsymbol{E}}^x\left[\tilde{\boldsymbol{E}}^{\hat{X}(\tilde{\tau})}\left[w(\tilde{X}(t))\right]\right]$, where $\hat{w}$ is the extension of w to E_∂ vanishing in $\mathsf{E} \setminus \mathsf{E}_M$.
■

According to the above theorems, we propose the following argument.

Theorem 5.12. *Suppose that Assumption 7 holds. For $a > 0$ and $f, g \in B_0^\partial(\mathsf{E}_\partial)$, we have $V_X^M(x) = V_{\hat{X}}(x) = V_{\tilde{X}}(x)$ for all $x \in \mathsf{E}_M$. Additionally, $V^M(x) = \hat{V}(x) = 0$ holds for $x \in \mathsf{E}_\partial \setminus \mathsf{E}_M$.*

Proof. First, we prove $V_X^M(x) = V_{\hat{X}}(x)$ for $x \in \mathsf{E}$. On the one hand, let $x \in \mathsf{E}_\partial \setminus \mathsf{E}_M$. Since $M(t)(\omega) = 0$ for all $t \ge 0$, it is obvious that $V_X^M(x) = 0$. Additionally, $\hat{X}(t) = \partial$ for all $t \ge 0$ $\hat{\boldsymbol{P}}^x$ almost surely. Since $f(\partial) = g(\partial) = 0$ by $f, g \in B_0(\mathsf{E}_\partial)$, $\hat{V}(x) = 0$ so that $V_X^M(x) = V_{\hat{X}}(x)$.

On the other hand, Let $x \in \mathsf{E}_M$. We first prove that $V_X^M(x) \ge V_{\hat{X}}(x)$. For each $\hat{\tau} \in \hat{\mathfrak{T}}$, we can define a stopping time $\tau \in \mathfrak{T}$ satisfying (5.2.7), whose existence has been shown in Theorem 5.8. Therefore, by (5.2.9) and (5.2.10), for any $x \in \mathsf{E}_M$, for any $\hat{\tau} \in \hat{\mathfrak{T}}$, there exists $\tau \in \mathfrak{T}$ such that

$$\begin{aligned}
&\boldsymbol{E}^x\left[\int_0^\tau e^{-as}M(s)f(X(s))\mathrm{d}s + e^{-a\tau}M(\tau)g(X(\tau))\right]\\
=&\hat{\boldsymbol{E}}^x\left[\int_0^{\hat{\tau}} e^{-as}f(\hat{X}(s))\mathrm{d}s + e^{-a\hat{\tau}}g(\hat{X}(\hat{\tau}))\right],
\end{aligned}$$

that is, the value function $V_X^M(x) \ge V_{\hat{X}}(x)$.

Conversely, it also can be proved that, for any $\tau \in \mathfrak{T}$, there exists $\hat{\tau}(\hat{\omega}) := \tau(\pi(\hat{\omega})) \wedge \gamma(\hat{\omega}) \in \hat{\mathfrak{T}}$ such that the above equation is also satisfied similarly, that is, $V_X^M(x) \leq V_{\hat{X}}(x)$ for all $x \in \mathsf{E}_M$. Then, $V_X^M(x) = V_{\hat{X}}(x)$ holds for all $x \in \mathsf{E}$.

Next, we only need to prove that $\hat{V}(x) = \tilde{V}(x)$ for $x \in \mathsf{E}_M$. By Theorem 5.11, for any stopping time $\tilde{\tau} \in \tilde{\mathfrak{T}}$, there exists a stopping time $\hat{\tau} \in \hat{\mathfrak{T}}$ such that $\tilde{\tau} = \hat{\tau}$ on $\tilde{\Omega}$, then for any $x \in \mathsf{E}_M$,

$$\begin{aligned}
&\hat{\boldsymbol{E}}^x\left[\int_0^{\hat{\tau}} e^{-as} f(\hat{X}(s))\mathrm{d}s + e^{-a\hat{\tau}} g(\hat{X}(\hat{\tau}))\right] \\
=&\tilde{\boldsymbol{E}}^x\left[\int_0^{\tilde{\tau}} e^{-as} f(\tilde{X}(s))\mathrm{d}s + e^{-a\tilde{\tau}} g(\tilde{X}(\tilde{\tau}))\right].
\end{aligned}$$

Therefore, for $x \in \mathsf{E}_M$, $\hat{V}(x) \leq \tilde{V}(x)$. Conversely, for any stopping time $\hat{\tau} \in \hat{\mathfrak{T}}$, we have $\tilde{\tau}(\tilde{\omega}) := \hat{\tau}(\tilde{(\omega)})$ that is $\tilde{\tau} \in \tilde{\mathfrak{T}}$. Again, the above equation also implies $\hat{V}(x) \geq \tilde{V}(x)$ for all $x \in \mathsf{E}_M$. Then, the proof is completed. ■

5.3 Main Theorems

Let ∂ be a point not in E. Define the open subsets of E_∂ as all the open subsets of E and the set $V = (\mathsf{E}_M \setminus K) \cup \{\partial\}$, where K is closed and compact in E. Then, since $\mathsf{E}_M \in \mathcal{E}$ and then $\mathsf{E}_M \setminus K \in \mathcal{E}$, the Borel σ-algebra of E_∂ is $\mathcal{E}_\partial$ which is the σ-algebra of E_∂ generated by $\mathcal{E}$. Hence, this construction of the point ∂ will not contradict any statement in the last section. Moreover, by one point compactification, we also know that $\mathsf{E}_M^\partial := \mathsf{E}_M \cup \partial$ is compact and E_M is dense in E_M^∂ if E_M is not compact. Therefore, for any $w \in \mathcal{C}_0(\mathsf{E}_M)$, there exists a continuous extension $\tilde{w} \in \mathcal{C}_0(\mathsf{E}_M^\partial)$ defined as

$$\tilde{w}(x) = \begin{cases} w(x) & x \in \mathsf{E}_M, \\ 0 & \text{otherwise} . \end{cases} \tag{5.3.1}$$

Remark 5.13. *Given $w \in \mathcal{C}_0(\mathsf{E}_M)$, the extension $\tilde{w}$ to E_∂ by letting $\bar{w}(x) = 0$ for $x \notin \mathsf{E}_M^\partial$ may not be also continuous. For example, $\bar{w}$ is discontinuous at* 0 *when $E = \mathbb{R}$, $\mathsf{E}_M = [0,1)$ and $w(x) = 1 - x$ for $x \in [0,1)$. However, $\bar{w}$ is still a bounded Borel measurable function.*

In what follows, we will show that $\tilde{X}$ is a Feller process under some appropriate conditions. Define the transition semigroup of $\tilde{X}$ on $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ on $B(\mathsf{E}_M^\partial)$ by

$$\tilde{\mathcal{P}}_t w(x) := \tilde{\boldsymbol{E}}^x\Big[w(\tilde{X}(t))\Big] \text{ for } x \in \mathsf{E}_M^\partial, t \geq 0, w \in B(\mathsf{E}_M^\partial). \tag{5.3.2}$$

Theorem 5.14. *Under Assumption 7 and Assumption 8, $\tilde{X}$ is a Feller process with the state space $(\mathsf{E}_M^\partial, \mathcal{E}_M^\partial)$.*

Proof. Let $t \geq 0$ and $w \in \mathcal{C}(\mathsf{E}_M^\partial)$. We first prove $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$ has the Feller property that is $\tilde{\mathcal{P}}_t w \in \mathcal{C}(\mathsf{E}_M^\partial)$. Since $\boldsymbol{P}^x$ is a probability measure for $x \in \mathsf{E}_M^\partial$ by (5.2.12), we have $\tilde{\mathcal{P}}_t(w - w(\partial)\mathbf{1}) = \tilde{\mathcal{P}}_t w - w(\partial)\mathbf{1}$. Hence, $\tilde{\mathcal{P}}_t w \in \mathcal{C}(\mathsf{E}_M^\partial)$ is equivalent with $\tilde{\mathcal{P}}_t(w - w(\partial)\mathbf{1}) \in C(\mathsf{E}_M^\partial)$. Therefore, we can simply set $w(\partial) = 0$ in what follows with losing generality.

By Theorem 5.7 and Theorem 5.10,

$$\tilde{\mathcal{P}}_t w(x) = \boldsymbol{E}^x\Big[M(t)\tilde{w}(X(t)); X(t) \in \mathsf{E}\Big] \text{ for } x \in \mathsf{E}_\partial, \tag{5.3.3}$$

where $\tilde{w}$ is the extension of w by setting $\tilde{w}(x) = 0$ for all $x \in \mathsf{E}_\partial \setminus \mathsf{E}_M^\partial$. Since we have set $w(\partial) = 0$, $\tilde{w}|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$. By Assumption 8, we have $(\tilde{\mathcal{P}}_t w)|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$. It implies that that $\tilde{\mathcal{P}}_t w \in \mathcal{C}(\mathsf{E}_M^\partial)$ since $\tilde{\mathcal{P}}_t w(\partial) = 0$ by (5.3.3). Then, the Feller property is shown.

Moreover, since X and M are both right continuous, $\tilde{X}$ is also right continuous. [Kallenberg, 2006, Lemma 17.3] implies $\tilde{\mathcal{P}}_t w(x) \to$ as $t \to 0$ for $x \in \mathsf{E}_M^\partial$ and $w \in \mathcal{C}(\mathsf{E}_M^\partial)$, since E_M^∂ is compact and metrizable. Hence, using [Böttcher et al., 2013, Lemma 1.4], it implies the strongly continuous property of $\{\tilde{\mathcal{P}}_t\}_{t\geq 0}$. ■

Corollary 5.15. *$\{\mathcal{Q}_t\}_{t\geq 0}$ is a Feller semigroup on $\mathcal{C}_0(\mathsf{E}_M)$.*

We first give the properties of V_X^M defined by Definition 5.1.3 and the optimal stopping time strategy.

Theorem 5.16. *$V_X^M|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$. The optimal stopping time strategy is given by*

$$\tau^* := \inf\{t \geq 0; V_X^M(X(t)) = g(X(t)) \text{ or } X(t) \notin \mathsf{E}_M\}. \tag{5.3.4}$$

Proof. This is directly from the construction of the stopping time by Theorem 5.8 and Theorem 5.12. ■

Then, we show the value function is the unique visocisty solution. Let $(\mathcal{G}, D(\mathcal{G}))$ be a core of the generator $(\mathcal{L}, D(\mathcal{L}))$ of $\{Q_t\}_{t\geq 0}$. Define

$$D(\mathcal{G}^*) := \{v \in \mathcal{C}_0(\mathsf{E}_M); v - \tilde{v}\mathbf{1} \in D(\mathcal{G})\}, \tag{5.3.5}$$

$$\mathcal{G}^* w := \mathcal{G}(w - \tilde{w}(\partial)\mathbf{1}) + \tilde{w}(\partial)\mathbf{1} \text{ for } w \in D(\mathcal{G}^*). \tag{5.3.6}$$

Theorem 5.17. *Under Assumption 7 and Assumption 8, the value function $V_X^M|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E}_M)$ defined by (5.1.3) is the unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to*

$$(aw - \mathcal{G}^* w - f|_{\mathsf{E}_M}, w - g|_{\mathsf{E}_M}) = 0. \tag{5.3.7}$$

Proof. We prove this theorem in three steps: In the first step we use Theorem 5.7 and Theorem 5.10 to transform our normal Markov process X to two relevant Markov processes $\hat{X}$ and $\tilde{X}$, which will be used for the proof of the theorem. In the second step, we construct the transition semigroup of the process $\tilde{X}$ and prove it is a Feller process. In the third step, we verify that Assumption 7 are satisfied in order to apply Theorem 3.17 to the process $\tilde{X}$ and then we conclude that there exists a unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (5.3.7), which coincides with $V|_{\mathsf{E}_M}$ by Theorem 5.12.

Step 1. It follows from Theorem 5.7 that we can construct a Markov process $\hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\boldsymbol{P}}^x)$ on the state space $(\mathsf{E}_\partial, \mathcal{E}_\partial)$ satisfying

$$\hat{\boldsymbol{E}}^x\big[w(\hat{X}(t))\big] = \boldsymbol{E}^x\big[w(X(t))M(t)\big], \tag{5.3.8}$$

for $t \geq 0$, $x \in \mathsf{E}$ and $w \in B_0^\partial(\mathsf{E}_\partial)$, where $B_0^\partial(\mathsf{E}_\partial) := \{v \in B(\mathsf{E}_\partial); v(\partial) = 0\}$. Define $\mathsf{E}_M^\partial := \mathsf{E}_M \cup \partial$ and let $\mathcal{E}_M^\partial$ be the trace of $\mathcal{E}_\partial$ on E_M^∂. Hence, by condition (3) in Assumption 7, one can use Theorem 5.10 to construct a normal Markov process $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}(t), \tilde{\theta}_t, \tilde{\boldsymbol{P}}^x)$ on the state space $(\mathsf{E}_M^\partial, \mathcal{E}_M^\partial)$, which is the restriction of $\hat{X}$ on E_M^∂. In addition, $\hat{X}$ and $\tilde{X}$ satisfy

$$\tilde{\boldsymbol{E}}^x\big[w(\tilde{X}(t))\big] = \hat{\boldsymbol{E}}^x\big[\bar{w}(\hat{X}(t))\big] \tag{5.3.9}$$

for $t \geq 0$, $x \in \mathsf{E}_M^\partial$ and $w \in B(\mathsf{E}_M^\partial)$, where $\bar{w}$ is the extension of w to E_∂ vanishing in $\mathsf{E}_\partial \setminus \mathsf{E}_M$.

Step 2. In this step, we will prove that the transition semigroup of the normal Markov process $\tilde{X}$ is equivalent to the semigroup $\{\tilde{\mathcal{P}}_t^M\}_{t\geq 0}$ defined by

$$\tilde{\mathcal{P}}_t^M w := \widehat{\mathcal{P}_t^M(w|_{\mathsf{E}_M} - w(\partial))} + w(\partial) \text{ for } t \geq 0, x \in \mathsf{E}_M^\partial \text{ and } w \in B(\mathsf{E}_M^\partial). \tag{5.3.10}$$

Let $w \in B(\mathsf{E}_M^\partial)$. Define $w_\partial := w - w(\partial) \in B_0^\partial(\mathsf{E}_M^\partial)$ and let $\bar{w}_\partial$ as the extension of w_∂ to E_∂ vanishing in $\mathsf{E}_\partial \setminus \mathsf{E}_M$. Then, $\bar{w}_\partial \in B_0^\partial(\mathsf{E}_\partial)$. Let $t \geq 0$ and $x \in \mathsf{E}_M \subseteq \mathsf{E}$. Then, by $w_\partial \in B_0^\partial(\mathsf{E}_\partial)$ and (5.3.8), we have for $x \in \mathsf{E}_M^\partial$

$$\tilde{\boldsymbol{E}}^x[w(\tilde{X}(t))] = \tilde{\boldsymbol{E}}^x[w_\partial(\tilde{X(t)})] + w(\partial) = \hat{\boldsymbol{E}}^x[\bar{w}_\partial(\hat{X}(t))] + w(\partial) \tag{5.3.11}$$

Thus, by $\bar{w}_\partial \in B_0^\partial(\mathsf{E}_\partial)$, (5.3.9), we have for $x \in \mathsf{E}_M$

$$\hat{\boldsymbol{E}}^x[\bar{w}_\partial(\hat{X}(t))] = \boldsymbol{E}^x[\bar{w}_\partial|_{\mathsf{E}}(X(t))M(t)] = Q_t\bar{w}_\partial|_{\mathsf{E}}(x) = \mathcal{P}_t^M w_\partial|_{\mathsf{E}_M}(x). \tag{5.3.12}$$

Hence, by (5.3.10), (5.3.11) and (5.3.12), we have $\tilde{\boldsymbol{E}}^x[w(\tilde{X}(t))] = \tilde{\mathcal{P}}_t^M w(x)$ for $x \in \mathsf{E}_M$. In addition, for $x = \partial$, since $\hat{\boldsymbol{E}}^\partial[\bar{w}_\partial(\hat{X}(t))] = 0$, by (5.3.11), we also have $\tilde{\boldsymbol{E}}^x[w(\tilde{X}(t))] = \tilde{\mathcal{P}}_t^M w(x)$.

Therefore, by condition (2) in Assumption 7, since $\{\mathcal{P}_t^M\}_{t\geq 0}$ is a Feller semigroup, $\tilde{X}(t)$ is a Feller process whose transition semigroup is equivalent with a Feller semigroup $\{\tilde{\mathcal{P}}_t^M\}_{t\geq 0}$.

Step 3. In order to apply Theorem 3.17 , we need prove that all the conditions of Assumption 1 in Chapter 3 with respect to $\tilde{X}$ are satisfied. By condition (3) in Assumption 7, it follows from Theorem 5.11 that $\tilde{X}$ is a strong Markov process. Let E_M^∂ be the one point compactification of E_M. Then, using Theorem 3.17 in Chapter 3, the value function $\tilde{V} \in \mathcal{C}(\mathsf{E}_M^\partial)$ is the unique viscosity solution associated with $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ to

$$\min(aw - \tilde{\mathcal{L}}w - \tilde{f}, w - \tilde{g}) = 0,$$

where $(\tilde{\mathcal{L}}, D(\tilde{\mathcal{L}}))$ is the generator of $\{\tilde{\mathcal{P}}_t^M\}_{t\geq 0}$ and $\tilde{f}, \tilde{g}$ are the extension of $f|_{\mathsf{E}_M}$ and $g|_{\mathsf{E}_M}$ to E_M^∂ respectively. Moreover, since $\{\tilde{\mathcal{P}}_t^M\}_{t\geq 0}$ is obtained from the Feller semigroup $\{\mathcal{P}_t^M\}_{t\geq 0}$ after one point compactification, by Proposition 3.24 in Chapter 3.24, we have $\tilde{V}|_{\mathsf{E}_M}$ is the unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ to (5.3.7). Using Theorem 5.12, we have $V^M|_{\mathsf{E}_M} = \tilde{V}|_{\mathsf{E}_M}$. This proof is completed. ■

5.4 Applications

In this section, we apply the result obtained in Theorem 5.17 to study several useful cases of the optimal stopping problems pertaining with multiplicative functionals.

5.4.1 Feller process with killing rate

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X(t), \theta_t, P^x)$ with the state space $(\mathsf{E}, \mathcal{E})$ and

$$M(t) = \exp\bigg(-\int_0^t v(X(s))\mathrm{d}s\bigg), \tag{5.4.1}$$

where v is a positive real valued function on E.

Proposition 5.18. *Given a strong Markov process X and a positive real value function v on E, let*

$$\mathcal{P}_t^v w(x) := \boldsymbol{E}^x\big[e^{-\int_0^t v(X(s))\mathrm{d}s} w(X(t))\big] \text{ for } w \in B(\mathsf{E}), x \in \mathsf{E} \text{ and } t \geq 0. \tag{5.4.2}$$

Assume that $\{\mathcal{P}_t^v\}_{t\geq 0}$ is a Feller semigroup on $\mathcal{C}_0(\mathsf{E})$ with a core operator $(\mathcal{G}_v, D(\mathcal{G}_v))$.

If $f, g \in \mathcal{C}_0(\mathsf{E})$ and $a > 0$, then the value function V^v given by

$$V^v(x) = \sup_\tau \boldsymbol{E}^x\left[\int_0^\tau e^{-as-\int_0^s v(X_r)\mathrm{d}r} f(X(s))\mathrm{d}s - e^{-a\tau-\int_0^\tau v(X_r)\mathrm{d}r} g(X(\tau))\right] \text{ for } x \in \mathsf{E} \tag{5.4.3}$$

is the unique viscosity solution associated with $(\mathcal{G}_v^*, D(\mathcal{G}_v^*))$ defined by (3.4.5) to

$$\min(aw - \mathcal{G}_v^* w - f, w - g) = 0 \tag{5.4.4}$$

Proof. Since v is positive real value function, $M := \{M(t)\}_{t\geq 0}$ defined by (5.4.1) is a continuous multiplicative functional and $M(t) > 0$ for all $t \geq 0$. Since M is a regular multiplicative functional, Theorem 5.17 yields Proposition 5.18. ∎

Remark 5.19. *Conditions guarantying that $\{\mathcal{P}_t^v\}_{t\geq 0}$ is a strongly continuous semigroup can be found in Chung and Zhao [2012] called* abstract Kato condition, *e.g. $\lim_{t\to 0^+}\sup_{x\in\mathsf{E}}\int_0^t \mathcal{P}_t v(x) = 0$. However, the Feller property of $\{\mathcal{P}_t^v\}_{t\geq 0}$ has to be analyzed case by case usually. Chung and Zhao [2012] also shows that if $\mathcal{P}_t^v : B(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ (i.e strong Feller property), $\{\mathcal{P}_t^v\}_{t\geq 0}$ satisfies the Feller property.*

Next we give an examples of processes whose value function defined by (5.4.3) is the unique viscosity solution associated with $(\mathcal{G}^*, D(\mathcal{G}^*))$ as given in Proposition 5.18.

Example 5.20. *(Feller diffusion) Assume that $\mathsf{E} = \mathbb{R}^n$ and define the life time of X by $\xi = \{t \geq 0; X(t) = \partial\}$. Feller diffusion is a Feller process which has a continuous path $t \to X(t)(\omega)$ on $[0, \xi)$ whose domain of the generator contains $\mathcal{C}_c^\infty(\mathbb{R}^n)$. It can be proved (see [Kallenberg, 2006, Theorem 17.24]) that the restriction $(\mathcal{G}^{FD}, D(\mathcal{G}^{FD}))$ of the infinitesimal generator of the Feller diffusion X on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is of the form as follows*

$$\mathcal{G}^{FD} = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} - v(x), \tag{5.4.5}$$

where $a_{ij}, b_i, v \in \mathcal{C}(\mathbb{R}^n)$, $v \geq 0$ and $\{a_{ij}(x)\}_{i,j}$ is non-negative symmetric matrix for all $x \in \mathsf{E}$. Let X be a Feller process with the state space $\mathsf{E} = \mathbb{R}^n$ whose core operator $(\mathcal{G}_0, \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined by

$$\mathcal{G}_0 w := \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} \text{ for } w \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

For example, Lévy with continuous path when a *and* b *are constant. Assume that* $v \in \mathcal{C}_b(\mathbb{R}^n)$ *and* $v \geq 0$. *Then* $\{\mathcal{P}_t^v\}_{t\geq 0}$ *defined by (5.4.2) is a Feller semigroup with a core operator* $(\mathcal{G}_v, \mathcal{C}_0^\infty(\mathbb{R}^n))$ *(see [Applebaum, 2009, Theorem 6.7.9]) given by*

$$\mathcal{G}_v w := \mathcal{G}_0 w - v \cdot w \text{ for } w \in \mathcal{C}_0^\infty(\mathbb{R}^n). \tag{5.4.6}$$

Proposition 5.18 implies that the value function V *defined by (5.4.3) is the unique viscosity solution associated with* $(\mathcal{G}_v^*, \mathcal{C}_*^\infty(\mathbb{R}^n))$ *to*

$$\min(aw - \mathcal{G}_v^* w - f, w - g) = 0$$

where

$$\mathcal{G}_v^* w := \mathcal{G}_v w + \tilde{w}(\partial) \cdot v \quad \text{for } w \in \mathcal{C}_*^\infty(\mathbb{R}^n) \tag{5.4.7}$$

Remark 5.21. *It may be surprising that* $(\mathcal{G}_v^*, \mathcal{C}_*^\infty(\mathbb{R}^n))$ *defined by (5.4.7) is different with usual version* $(\mathcal{G}_v, \mathcal{C}_*^\infty(\mathbb{R}^n))$ *defined by (5.4.6). Actually, if* $w \in \mathcal{C}_0(\mathbb{R}^n)$ *is a viscosity solution associated with* $(\mathcal{G}_v, \mathcal{C}_*^\infty(\mathbb{R}^n))$, *then* w *is also a viscosity solution associated with* $(\mathcal{G}_v^*, \mathcal{C}_*^\infty(\mathbb{R}^n))$. *For example, if* $\phi \in \mathcal{C}_*^\infty(\mathbb{R}^n)$ *satisfies* $\phi - w$ *has a global minimum equal* 0 *at* x *in* E, *then we have* $\tilde{\phi}(\partial) \geq \tilde{w}(\partial) = 0$ *such that, by* $v(x) \geq 0$,

$$\begin{aligned} & \min(a\phi(x) - (\mathcal{G}_v \phi(x) + v(x)\tilde{\phi}(\partial)) - f(x), \phi(x) - g(x)) \\ \leq\ & \min(a\phi(x) - \mathcal{G}_v \phi(x) - f(x), \phi(x) - g(x)) \\ \leq\ & 0. \end{aligned}$$

5.4.2 Feller process killed in a strong terminal time ζ

A $\mathcal{F}_t$-stopping time ζ is a strong terminal time of X if

$$\zeta = \tau + \zeta \circ \theta_\tau,$$

for all $\mathcal{F}_t$-stopping time τ almost surely on $\{\zeta > \tau\}$. In this section, we consider a multiplicative function M^ζ defined by

$$M(t)^\zeta(\omega) = \mathbf{1}_{\zeta(\omega)<t} \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{5.4.8}$$

which is a regular multiplicative functional. (See [Blumenthal and Getoor, 2007, Page 124]). Hence, by Theorem 5.17, we get the following proposition.

Proposition 5.22. *Given a strong Markov process* X *and a strong terminal time* ζ, *define*

$$\mathcal{P}_t^\zeta w(x) := \boldsymbol{E}^x\big[w(X(t))\mathbf{1}_{t<\zeta}\big] \text{ for } w \in B(\mathsf{E}), x \in \mathsf{E} \text{ and } t \geq 0. \tag{5.4.9}$$

Assume that $\{\mathcal{P}_t^\zeta\}_{t\geq 0}$ is a Feller semgiroup on $\mathcal{C}_0(\mathsf{E}_M)$ with a core operator $(\tilde{\mathcal{G}}_\zeta, D(\tilde{\mathcal{G}}_\zeta))$. In addition, $\mathsf{E}_M \in \mathcal{E}$ is a locally compact Hausdorff space with suitable metric and τ_{E_M} is a $\mathcal{F}_t$-stopping time. If $f, g \in B(\mathsf{E})$ with $f|_{\mathsf{E}_M}, g|_{\mathsf{E}_M} \in \mathcal{C}_0(\mathsf{E})$ and $a > 0$, then the value function V^ζ given by

$$V^\zeta(x) := \sup_\tau \boldsymbol{E}^x\left[\int_0^{\tau\wedge\zeta} e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau} g(X(\tau))\mathbf{1}_{\tau<\zeta}\right] \text{ for } x \in \mathsf{E}_M \quad (5.4.10)$$

is the unique viscosity solution associated with $(\mathcal{G}_\zeta^*, D(\mathcal{G}_\zeta^*))$ defined by (3.4.5) to

$$\min(aw - \mathcal{G}_\zeta^* w - f, w - g) = 0 \quad (5.4.11)$$

Example 5.23. *(Diffusion on $(0,\infty)$ killed at 0) Let $\{X\}_{t\geq 0} = \{B_t\}_{t\geq 0}$ be a standard Brownian motion with strong Markov property with a $\mathcal{F}_t$-stopping time $\zeta = \tau_0 := \inf\{t > 0; X(t) \notin (0,\infty)\}$. In addition, $M^\zeta = \{\mathbf{1}_{\tau_0<t}\}_{t\geq 0}$ is a regular multiplicative functional. Since $\boldsymbol{P}^x[M_0^{\tau_0} = 1] = 1$ for $x \in (0,\infty)$ and $\boldsymbol{P}^x[M_0^{\tau_0} = 0]$ for $x \in (-\infty, 0]$, we have $\mathsf{E}_M = (0,\infty)$. Furthermore, it is well known that $\{\mathcal{P}_t^{\tau_0}\}_{t\geq 0}$ is a Feller semigroup on $\mathcal{C}_0((0,\infty))$ defined by (5.4.9) whose core operator is given by*

$$\mathcal{G}_{kill}^{BC} w(x) = \frac{1}{2} D^2 w(x)$$

for $x \in (0,\infty)$ and $w \in D(\mathcal{G}_{kill}^{BC}) := \{w \in \mathcal{C}_0^2((0,\infty));\ D^2 w \in \mathcal{C}_0((0,\infty))\}$. Therefore, if $f|_{(0,\infty)}, g|_{(0,\infty)} \in \mathcal{C}_0((0,\infty))$ and $a > 0$, by Proposition 5.22, the value function $V^{\tau_0} \in \mathcal{C}_0((0,\infty))$ defined by

$$V^{\tau_0}(x) = \sup_\tau \boldsymbol{E}^x\left[\int_0^{\tau\wedge\tau_0} e^{-as} f(X(s))\mathrm{d}s + e^{-a\tau} g(X(\tau))\mathbf{1}_{\tau<\tau_0}\right] \text{ for } x \in (0,\infty)$$

is the unique viscosity solution associated with $(\mathcal{G}_{kill}^{BC}, D_*(\mathcal{G}_{kill}^{BC}))$ to

$$\min(aw - \mathcal{G}_{kill}^{BC} w - f|_{(0,\infty)}, w - g|_{(0,\infty)}) = 0,$$

where $D_*(\mathcal{G}_{kill}^{BC}) = \{w \in \mathcal{C}^2((0,\infty)) \cap \mathcal{C}_*((0,\infty));\ D^2 w \in \mathcal{C}_0((0,\infty))\}$.

Remark 5.24. *More generally, define $\zeta = \tau_{\mathcal{O}} := \{t > 0; X(t) \notin \mathcal{O}\}$, where $\mathcal{O}$ is a bounded open subset of E. If X is a Feller process with strong Feller property and $\mathcal{O}$ is regular, i.e. $\boldsymbol{P}^x[\tau_{\mathcal{O}} = 0] = 1$ for any $x \in \partial\mathcal{O}$, then the semigroup $\{\mathcal{P}_t^{\tau_{\mathcal{O}}}\}_{t\geq 0}$ on $\mathcal{C}_0(\mathcal{O})$ defined by*

$$\mathcal{P}_t^{\tau_{\mathcal{O}}} w := \boldsymbol{E}^x[w(X(t))\mathbf{1}_{\tau_{\mathcal{O}}<t}]$$

is a Feller semigroup. The definition of the infinitesimal generator can be also found in [Baeumer et al., 2016, Lemma 2.3]. In particular, the above is an *example where $\mathcal{O} = (0, \infty)$ and X is a standard Brownian motion. In addition, [Kolokoltsov, 2011, Theorem 6.2.2 and 6.2.4] also introduce the conditions for Lévy type process for which $\{\mathcal{P}_t^{\tau_{\mathcal{O}}}\}_{t \geq 0}$ is a Feller semigroup without assuming the strong Feller property and openness of $\mathcal{O}$. Furthermore, (5.4.9) has been also applied in the optimal stopping problems. For example, Palczewski and Stettner [2011] assumes its Feller property and give some extension results when f, g could be discontinuous at boundary in some particular examples.*

Chapter 6

Concluding Remarks

This thesis has dedicated work into deriving viscosity solutions of optimal stopping problems for Feller processes. The motivation of the research stemmed from a willingness to generalise results under the framework of solving optimal stopping problems. Instead of duplicating traditional techniques based on second-order differential equations, we detect a common condition for all this kind of problems and are able to propose an integrated proof of the existence and uniqueness for their associated viscosity solutions. In this way, we have avoided repeating similar proofs as often appeared in literature.

Chapter 3 uses penalty method to show the existence and uniqueness of viscosity solutions. We did not assume the underlying process to be any specific process. We only address that it should satisfy the Feller property. Hence, our results could be employed for many applications either in verifying results already presented in existing literature or via a new modelling for similar problems by a transformation of a Feller semigroup.

However, in our setting, the payoff functions used in optimal stopping problems are required to be bounded, whereas in the past literature, most of them just were not imposed with this condition, e.g., a majority of researches rely on the condition with polynomial growth only. To extend our results, it is possible to use the proposed method for strongly continuous semigroups on weighted function space instead of Feller semigroups on $\mathcal{C}_0(\mathsf{E})$. Furthermore, one can also adopt the iterative optimal stopping methods especially for the impulse control problems where the respective payoff functions are usually unbounded.

Chapter 4 further broadens the use of iterative optimal stopping methods by imposing weakened assumptions. We found that it can not only be employed for impulse control problems, but also assists in deriving value functions of a wider range of optimal stopping problems which are not limited to the standard form as discussed in Chapter 3.

Chapter 5 is trying to construct new optimal stopping problems with multiplicative functionals for nonconservative Feller semigroups. It can be seen as a complementary element of Chapter 3 whose Feller processes have infinite life time. However, in this thesis we did not present many valuable applications. For instance, the doubly Feller process which satisfies both strong Feller property and Feller property will be worth studying in the future. Furthermore, the payoff functions in this chapter have to be vanishing at the boundary, which is actually a more restricted condition compared with the previous two chapters. Future work would be plausible focusing on removing such kind of restriction.

Another possible branch of application of the results demonstrated in this thesis is for optimal stopping problems whose underlying process has a state space not limited to $\mathbb{R}^n$. For example, applications can be carried out for a stochastic process generated by the delayed stochastic differential equations whose state space is continuous functions on $[0, T]$. A second example would be a branching process with an infinite dimensional state space.

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